On Augmentations of Legendrian Knots

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Abstract

We study knots, which are embeddings of the circle into $\mathbb{R}^3$. We require that the knot lie tangent to a "plane field," induced by the standard contact structure of $\mathbb{R}^3$. Such a knot is called Legendrian. By building an algebraic structure from crossings and paths on Legendrian knots, we can extract information in hopes of building an invariant. However, this algebraic structure is usually quite complicated; a solution to this is to investigate maps from this structure to simpler ones. In this project we study maps to the integers mod n and to matrix rings.

Introduction

- A topological knot is an embedding, or continuous injection with continuous inverse on its image, of the circle into $\mathbb{R}^3$.
- The basic problems of knot theory include classification and differentiation between knots. We say that two knots are the same if they can be continuously deformed into one another while at every instant remaining an embedding. This is called isotopy.
- Legendrian knot theory is the study of knots in $\mathbb{R}^3$ which lay tangent to the plane field in the standard contact structure of $\mathbb{R}^3$. Thus, a Legendrian knot $\Lambda(\theta) = (x(\theta), y(\theta), z(\theta))$ must satisfy

$$z'(\theta) = y(\theta)x'(\theta).$$

- In Legendrian knot theory, knots are classified up to Legendrian isotopy, meaning that knots must be smoothly deformed into one another and at every instant in the deformation, the deformed knot is Legendrian. This is a finer classification of knots, as seen in the below figure.

Figure 1: The Chekanov examples of two knots that are continuously and smoothly isotopic, but not Legendrian isotopic.

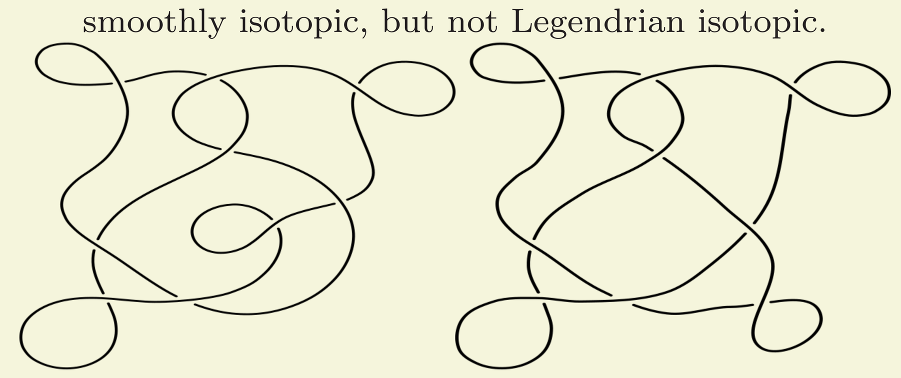
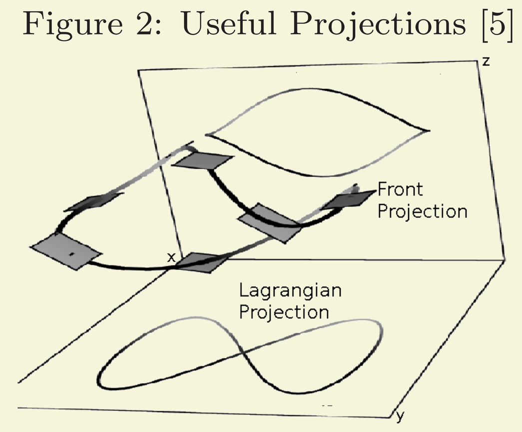


Figure 2: Useful Projections [5]



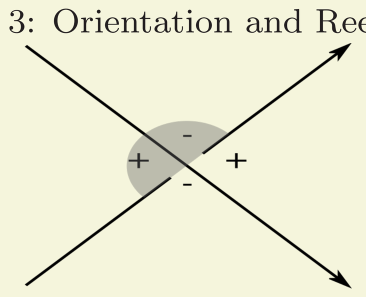
- We study Legendrian knots in two projections: the **front** (xz) projection and the **Lagrangian** (xy) projection.

Chekanov-Eliashberg DGA

The Checkanov-Eliashberg Differential Graded Algebra (DGA) is a Legendrian knot invariant constructed from the Lagrangian diagram of a Legendrian knot.

- The algebra is given by words in $q_i \in Q$ which correspond to crossings on the Lagrangian diagram.
- We set the grading $|q_i|$ to be the Maslov index at that point, and define $|q_i q_j| = |q_i| + |q_j|$.
- We define the **Reeb sign** of a corner by the signs in the diagram below, and the **orientation sign** of a corner of a positive intersection which is grey to be negative.

Figure 3: Orientation and Reeb signs

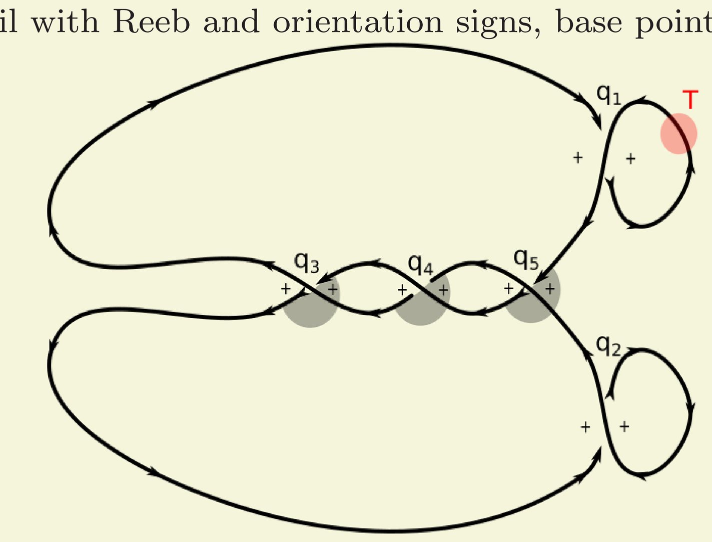


- The differential is given by immersed polygons in the knot, along with the condition that $\partial \circ \partial = 0$.

An Example – The Trefoil

First, we choose a base point and an orientation, followed by labeling Reeb and orientation signs.

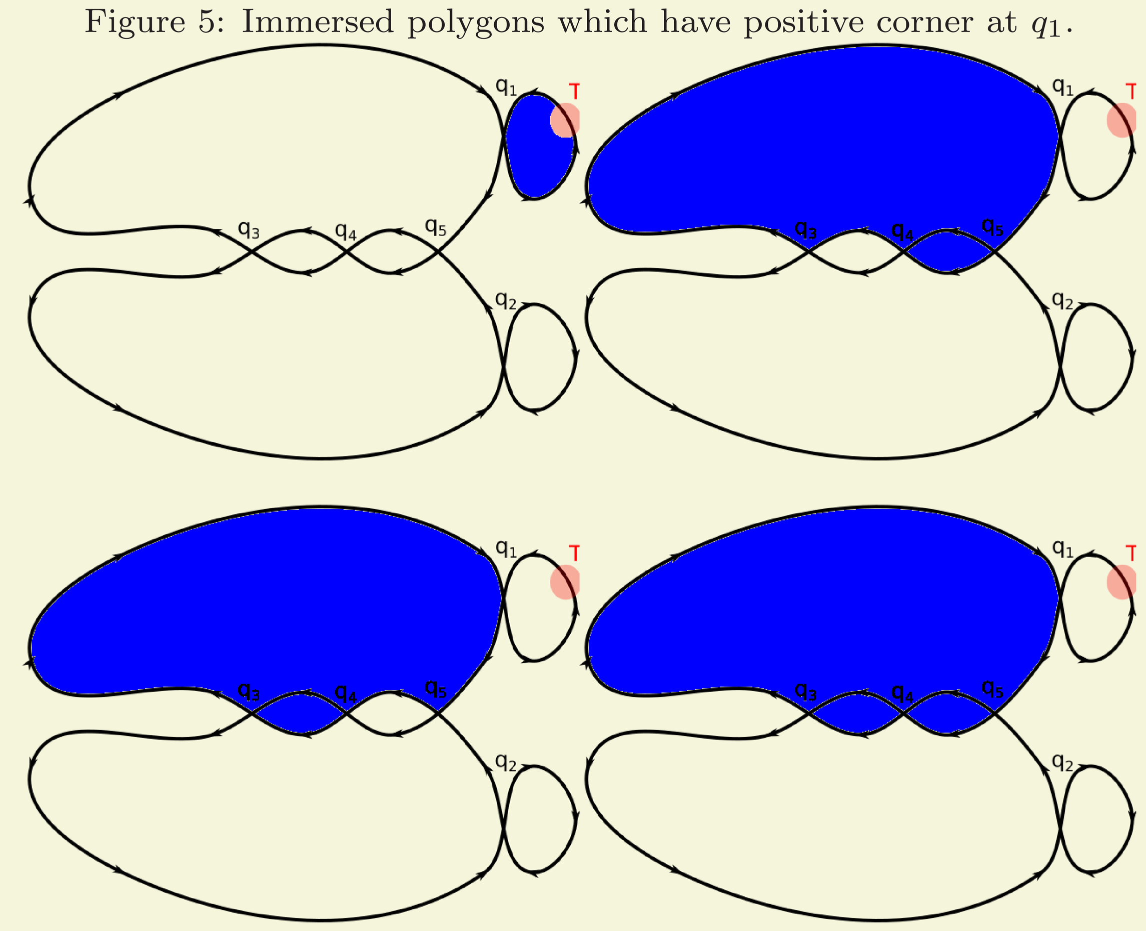
Figure 4: Trefoil with Reeb and orientation signs, base point, and orientation.



The DGA of the trefoil is thus $\mathbb{Z}\langle q_1, q_2, q_3, q_4, q_5, T\rangle$. We calculate the gradings of terms via their Maslov indexes.

$$|q_1| = |q_2| = 1 \qquad \text{and} \qquad |q_3| = |q_4| = |q_5| = 0.$$

Figure 5: Immersed polygons which have positive corner at q_1.



Lastly, we examine the immersed polygons in the above figure, finding

$$\partial q_1 = T + q_3 + q_5 + q_3 q_4 q_5$$
$$\partial q_2 = 1 - q_3 - q_5 - q_5 q_4 q_3$$
$$\partial q_3 = \partial q_4 = \partial q_5 = 0.$$

Augmentations

- Let R be a unital ring. An **augmentation** of a DGA $\mathcal{A}$ is an algebra map $\epsilon : \mathcal{A} \to R$ so that $\epsilon(1) = 1$. An augmentation is called graded if it has support on elements with degree 0.
- The existence of augmentations is a Legendrian invariant, and augmentations are used in other invariants to extract information from the DGA.
- In the example above, there are several augmentations to $\mathbb{Z}/2$.

Augmentations from the trefoil to $\mathbb{Z}/2$

ϵ_i	$\epsilon_i(T)$	$\epsilon_i(q_1)$	$\epsilon_i(q_2)$	$\epsilon_i(q_3)$	$\epsilon_i(q_4)$	$\epsilon_i(q_5)$
ϵ_1	1	0	0	0	0	1
ϵ_2	1	0	0	1	0	0
ϵ_3	1	0	0	0	1	1
ϵ_4	1	0	0	1	1	0
ϵ_5	1	0	0	1	1	1

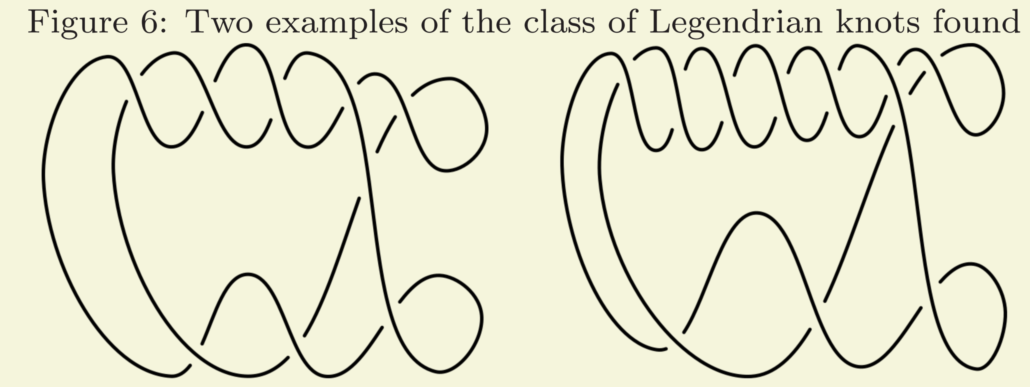
Goals

- Find a class of Legendrian knots for which there exists and augmentation $\epsilon : \mathcal{A} \to \mathbb{Z}/n$ such that $\epsilon(T) \neq \pm 1$.
- Does it make a difference which ring our augmentations map to, or what grading we put on the DGA?
- Can we construct invariants based on the image of these augmentations?

Results

- We found a class of Legendrian knots for which every knot has an augmentation to $\mathbb{Z}/n$ such that $\epsilon(T) \neq \pm 1$. By adding more crossings in the upper part of the knot over and over, we achieve a class of knots.

Figure 6: Two examples of the class of Legendrian knots found



- We have found many augmentations $\epsilon : \mathcal{A} \to GL_2(R)$.

Future Work

- Different Legendrian knots have augmentations to $\mathbb{Z}/n$ and $GL_2(\mathbb{Z}/n)$ for different n. Can an invariant be constructed out of this?
- Finding augmentations to $GL_2(R)$ involves solving a series of nonlinear matrix equations. Are there methods we could employ besides brute force in order to expedite the search for augmentations?
- Are there certain classes of knots for which we can show that all augmentations must send T to $\pm 1 \in \mathbb{Z}$ or $\pm I \in GL_2(\mathbb{Z}/n)$?
- Does there exist two knots which cannot be distinguished by augmentations to $\mathbb{Z}/n$ but can be distinguished by augmentations to $GL_2(\mathbb{Z}/n)$?

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