Differential Equations Comprehensive Exam Questions

1. We consider the differential equation in which we make the parameter change with time in a prescribed manner. That is, for a given C^∞ function $u : \mathbb{R} \to \mathbb{R}$, $\epsilon \in \mathbb{R}$, we consider the equation
$$y' = f(y, \epsilon u(t))$$
Denote by $\Phi_t(y_0; \epsilon)$ the solution that has initial conditions $y(0) = y_0$.

   Compute $\frac{d}{d\epsilon}\Phi_t(y_0, \epsilon)|_{\epsilon=0}$ in terms of the solutions of the variational equation.

   We recall that $M_t(y_0)$, the solutions of the variational equations are
$$M_t(y_0) = D_{y_0}\Phi_t(y_0; 0))$$

   Solution:

   There are several solutions to the problem. You can add an extra variable ϵ which satisfies $\epsilon' = 0$ and another variable t which satisfies $t' = 1$.

   We then obtain that
$$D_\epsilon \Phi_t(y_0; \epsilon)|_{\epsilon=0} = \int_0^t M(t-s)u(s)\, ds$$

2. Consider the linear autonomous differential equation:
$$y' = A(t)y + f(t)$$
Show that if $A(t)$ is an antisymmetric matrix and $\int_0^\infty |f(t)|\, dt < \infty$, then, all the solutions are bounded.

   Hint: Do first the case when $f(t)$ is not present.

   Bonus: Show that the same conclusion is true even if we allow the matrix A to depend also on y, but remaining antisymmetric.

   Solution: There are several solutions:

   The linear case can be solved observing that U_s^t, the evolution operator for $y' = A(t)y$ is an isometry and that, using the variation of parameters formula, we have $y(t) = \int_0^t U_s^t f(s)\, ds$.

To solve the non-linear case – which implies the linear case –, we compute:

$$\frac{d}{dt}|y|^2 = 2y \cdot y' = 2A(t,y)y \cdot y + f \cdot y = f \cdot y$$

where we have used the antisymmetry of $A(t)$. Hence, using Cauchy-Schwartz inequality, we obtain:

$$\Big|\frac{d}{dt}|y|^2\Big| \leq |f|(|y|^2)^{1/2}$$

$$\Big|(|y|^2)^{-1/2}\frac{d}{dt}|y|^2\Big| \leq |f|$$

Integrating:

$$2|y(t)| = 2(|y(t)|^2)^{1/2} \leq 2|y(0)| + \int_0^t |f(s)|\, ds$$

3. Fix a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^n$ of Lipschitz constant M (i.e. $|F(x) - F(y)| \leq M|x - y|$).

   Form the Picard operator acting on $C^0(\mathbb{R}^+, \mathbb{R}^n)$, the space of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^n$ endowed with the supremum norm

$$\mathcal{F}[y](t) = a + \int_0^t F(y(s))\, ds$$

   Show that for any pair of functions $y, z \in C^0(\mathbb{R}^+, \mathbb{R}^n)$ we have:

$$|\mathcal{F}^n[y](t) - \mathcal{F}^n[z](t)| \leq M^n|t|^n/n!$$

   where $\mathcal{F}^n$ means apply the operator n times.

Solution: We denote by $y_n(t) = \mathcal{F}^n[y](t)$ $z_n(t) = \mathcal{F}^n[z](t)$.

We note that the desired inequality is true for $n = 0$.

We assume that it is true for $n \leq N$. Then, we have

$$|\mathcal{F}^{N+1}[y](t) - \mathcal{F}^{N+1}[z](t)| \leq \int_0^t |F(y_N(s)) - F(z_N(s))|\, ds \leq \int_0^t M|y_N(s) - z_N(s)|\, ds$$
$$\leq \int_0^t MM^N s^N/N!\, ds$$

4. Solve the equation for the unknown $z = z(x, y)$

$$x(y - z)z_x + y(z - x)z_y = z(x - y) \tag{1}$$

with the condition

$$z = x \quad \text{along} \quad y = \frac{2x}{x^2 - 1}, \ |x| < 1. \tag{2}$$

Specify the domain of the solution carefully.

Solution: The characteristic of (1) is given by the equations

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

or equivalently a curve $(x(s), y(s), z(s))$ parametrized by s is a characteristic if it solve the ODE system

$$x_s = x(y - z), \quad y_s = y(z - x), \quad z_s = z(x - y). \tag{3}$$

Write condition (2) as a curve given by the parametric equation

$$P(\tau) = \big(x(\tau), y(\tau), z(\tau)\big) = \big(\tau, \frac{2\tau}{\tau^2 - 1}, \tau\big), \quad |tau| < 1.$$

Its tangent vector is

$$P_\tau(\tau) = \big(1, -2\frac{\tau^2 + 1}{(\tau^2 - 1)^2}, 1\big).$$

Its second component $y_\tau(\tau) \neq 0$ for all $|\tau| < 1$ while the y_s component of the vector field given in (3) is always 0. Therefore condition (2) is non-characteristic and a local solution exists uniquely.

One observes that the ODE system has two first integrals

$$u(x, y, z) = x + y + z, \quad v(x, y, z) = xyz$$

and $u = v$ along the curve given by (2). Therefore, the first integral

$$w(x, y, z) = u - v = x + y + z - xyz$$

of (3) satisfies that its zero level surface 1.) contains the curve defined by (2) and 2.) is tangent to characteristic vector field given (3). We obtain that the graph of the solution $z(x, y)$ must be contained in the zero level surface of $w(x, y, z)$. Namely, the solution is given by

$$z(x, y) = \frac{x + y}{xy - 1}.$$

Since along the curve (2) it holds $xy = \frac{2x^2}{x^2 - 1} < 1$, the domain of the solution $z(x, y)$ is given by $xy < 1$.

5. Solve the following initial boundary value problem

$$u_t + (u^2)_x = 0, \qquad x > 0,\ t > 0$$

under the conditions

$$u(0,x) = \frac{1}{2},\ x > 0; \qquad u(t,0) = \begin{cases} 1 & 0 < t < 1 \\ \frac{1}{2} & t > 1. \end{cases}$$

Solution: For $0 < t \le 4$ the solution is given by

$$u(t,x) = \begin{cases} \frac{1}{2} & 0 < x < t-1 \\ \frac{x}{2(t-1)} & t-1 \le x \le 2(t-1) \\ 1 & 2(t-1) < x < \frac{3t}{2} \\ \frac{1}{2} & x > \frac{3t}{2}. \end{cases}$$

The line $x = \frac{3t}{2}$ is a shock curve and we also have a rarefaction wave.

For $t > 4$ the solution is given by

$$u(t,x) = \begin{cases} \frac{1}{2} & 0 < x < t-1 \\ \frac{x}{2(t-1)} & t-1 \le x < (t-1) + \sqrt{3(t-1)} \\ \frac{1}{2} & x > (t-1) + \sqrt{3(t-1)}. \end{cases}$$

The curve $x = (t-1) + \sqrt{3(t-1)}$ is a shock curve.

6. Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. Let $f \in C(\mathbb{R}^n_+)$, $g \in C(\mathbb{R}^{n-1})$. Prove that there exists at most one bounded solution $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ of the boundary value problem

$$\Delta u = f \quad \text{in } \mathbb{R}^n_+; \qquad u = g \quad \text{if } x_n = 0.$$

Give a counterexample if u is not required to be bounded.

Solution: It is enough to prove that if $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$, is bounded, and solves

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n_+; \qquad u = 0 \quad \text{if } x_n = 0$$

then $u = 0$. We extend u to $\mathbb{R}^n$ by setting

$$\tilde{u}(x) = \begin{cases} u(x) & x_n \ge 0 \\ -u(x_1, ..., x_{n-1}, -x_n) & x_n < 0. \end{cases}$$

For $r > 0$, denote $B_r = \{x : |x| < r\}, B_r^+ = \{x : |x| < r, x_n > 0\}$. Let v be the solution of

$$\Delta v = 0 \quad \text{in } B_r; \qquad v = \tilde{u} \quad \text{on } \partial B_r.$$

Since $\tilde{v}(x) = -v(x_1, ..., x_{n-1}, -x_n)$ also satisfies

$$\Delta \tilde{v} = 0 \quad \text{in } B_r; \qquad \tilde{v} = \tilde{u} \quad \text{on } \partial B_r,$$

by uniqueness of solutions we obtain $\tilde{v} = v$ in B_r. (Alternatively one can use the fact that

$$\tilde{v}(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r} \frac{\tilde{u}(y)}{|x-y|^n} dS(y),$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$, to check directly that $v(x) = -v(x_1, ..., x_{n-1}, -x_n)$.) In particular we have $v(x_1, ..., x_{n-1}, 0) = 0$. Therefore $v = u$ on ∂B_r^+ so by uniqueness of solutions of

$$\Delta u = 0 \quad \text{in } B_r^+; \qquad u = \tilde{u} \quad \text{on } \partial B_r^+,$$

we have $u = v$ in B_r. This implies that $\tilde{u} = \tilde{v} = v$ in B_r. Thus u is harmonic in every ball B_r, and hence it is harmonic in $\mathbb{R}^n$. It now follows from Liouville's theorem that u is constant and thus $u = 0$.

When $g = 0$ any function $u(x) = cx_n$ solves the PDE.

7. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $u \in C^{1,2}([0,\infty) \times \bar{\Omega})$, $v \in C^2(\bar{\Omega})$ be solutions of

$$\begin{cases} u_t - \Delta u = 0, & x \in \Omega,\ t > 0 \\ u(0,x) = g(x), & x \in \Omega \\ u(t,x) = f(x), & x \in \partial\Omega,\ t > 0 \end{cases}$$

and

$$\Delta v = 0, \quad x \in \Omega; \qquad v(x) = f(x), \quad x \in \partial\Omega$$

respectively. Prove

$$\lim_{t \to +\infty} u(t,x) = v(x).$$

Solution: It is enough to prove that if $w \in C^{1,2}([0,\infty) \times \bar{\Omega})$ is a solution of

$$\begin{cases} w_t - \Delta w = 0, & x \in \Omega,\ t > 0 \\ w(0,x) = g(x), & x \in \Omega \\ w(t,x) = 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

then

$$\lim_{t\to+\infty} w(t,x) = 0.$$

Let $\Omega \subset B(0,R)$ for some $R > 0$ and let η be a constant such that $|g(x)| \le \eta - R^2$ for all $x \in \Omega$. Denote $\gamma = 2n/\eta$ and define

$$h(t,x) = e^{-\gamma t}(\eta - |x|^2).$$

Then for $\psi = w - h$ we have

$$\psi_t(t,x) - \Delta\psi(t,x) = e^{-\gamma t}(\gamma(\eta - |x|^2) - 2n) \le e^{-\gamma t}(\gamma\eta - 2n) = 0.$$

Moreover $\psi(0,x) = g(x) - (\eta - |x|^2) \le 0$ for $x \in \Omega$, and $\psi(t,x) \le 0$ for $x \in \partial\Omega, t > 0$. Therefore $\psi \in C^{1,2}([0,\infty) \times \bar{\Omega})$ satisfies

$$\begin{cases} \psi_t - \Delta\psi \le 0, & x \in \Omega,\ t > 0 \\ \psi(0,x) \le 0, & x \in \Omega \\ \psi(t,x) \le 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

It thus follows from the maximum principle that $\psi \le 0$. This implies that

$$w(t,x) \le e^{-\gamma t}(\eta - |x|^2) \le \eta e^{-\gamma t}.$$

Similarly we prove that

$$w(t,x) \ge -e^{-\gamma t}(\eta - |x|^2) \ge -\eta e^{-\gamma t}.$$

Thus w converges uniformly to 0 when $t \to +\infty$ at a rate $e^{-\gamma t}$.

8. Let $\Omega \subset \mathbb{R}^2$ be a domain and $u \in C^0(\Omega)$ satisfies the mean value property

$$u(x) = \frac{1}{2\pi r} \int_{\partial B(x,r)} u(x')\, dS_{x'} \tag{4}$$

for any $x \in \Omega$ and $r > 0$ such that the closed disk $\overline{B(x,r)} \subset \Omega$. Prove u is harmonic on Ω.

Solution: For any $x \in \Omega$ and $r > 0$ such that the closed disk $\overline{B(x,r)} \subset \Omega$, we first claim that the above assumption implies the mean value property

$$u(x) = \frac{1}{\pi r^2} \int_{B(x,r)} u(x')\, d\mu \tag{5}$$

where $d\mu$ is the Lebesgue measure. In fact, in polar coordinates

$$\int_{B(x,r)} u(x')\,d\mu = \int_0^r \int_0^{2\pi} u(r',\theta)r'd\theta dr'. \tag{6}$$

Notice

$$\int_0^{2\pi} u(r',\theta)r'd\theta = \int_{\partial B(x,r')} u(x')\,dS_{x'}.$$

Therefore (4) and (6) imply

$$\int_{B(x,r)} u(x')\,d\mu = \int_0^r 2\pi r' u(x) dr' = \pi r^2 u(x)$$

and thus (5) follows.

Let $x_0 \in \Omega$ be an arbitrary point. Take $r_0 > 0$ such that the closed disk $\overline{B(x_0,r_0)} \subset \Omega$, let $U(x')$, $x' \in \overline{B(x_0,r_0)}$, be the solution of the boundary value problem of the Laplace equation

$$\Delta U = 0 \text{ in } B(x_0,r_0), \quad \text{and} \quad U(x) = u(x) \text{ on } \partial B(x_0,r_0)$$

which can be explicitly written by using the Poisson integral formula and satisfies $U \in C^2\big(B(x_0,r_0)\big) \cap C^0\big(\overline{B(x_0,r_0)}\big)$. Let $v = u - U$. It is clear that $v \in C^0\big(\overline{B(x_0,r_0)}\big)$ also satisfies the mean property (5) on $\overline{B(x_0,r_0)}$ as both u and U do. An immediate consequence of (5) is that v satisfies the maximal principle on $\overline{B(x_0,r_0)}$. Since $v \equiv 0$ on $\partial B(x_0,r_0)$ due to its definition, we obtain $u = U$ on $B(x_0,r_0)$. Therefore $u \in C^2\big(B(x_0,r_0)\big)$ and $\Delta u = 0$ on $B(x_0,r_0)$. As $x_0 \in \Omega$ is arbitrarily taken, it follows that $u \in C^2(\Omega)$ and $\Delta u = 0$ on Ω.