Discrete Mathematics Comprehensive Exam Questions

1. Prove that for every integer $k \geq 1$ there exists an integer N such that if the subsets of $\{1, 2, \ldots, N\}$ are colored using k colors, then there exist disjoint non-empty sets $X, Y \subseteq \{1, 2, \ldots, N\}$ such that X, Y and $X \cup Y$ receive the same color.
Hint. You may want to consider intervals.

Solution: By Ramsey's theorem there exists an integer N such that for every k-coloring of 2-element subsets of $\{1, 2, \ldots, N+1\}$ there exists a 3-element set $A \subseteq \{1, 2, \ldots, N+1\}$ such that all 2-element subsets of A receive the same color. We claim that N satisfies the requirements of the problem. For $i, j \in \{1, 2, \ldots, N+1\}$ with $i < j$ we color the set $\{i, j\}$ using the color of the set $\{i, i+1, \ldots, j-1\}$. By the choice of N there exist $i, j, k \in \{1, 2, \ldots, N+1\}$ such that $i < j < k$ and the sets $\{i, j\}$, $\{j, k\}$ and $\{i, k\}$ receive the same color. Then the sets $X := \{i, i+1, \ldots, j-1\}$ and $Y := \{j, j+1, \ldots, k-1\}$ are as desired.

2. A proper list-coloring of a graph $G = (V, E)$ from lists $\{L_v \subset \mathbb{N} \mid v \in V\}$ is a function $c : V \to \mathbb{N}$ such that $c(v) \in L_v$ for all $v \in V$ and $c(u) \neq c(v)$ for all $\{u, v\} \in E$.

   Let r be a natural number. Prove that if for all $v \in V$ we have $|L_v| = 10r$ and for all $j \in L_v$ there are at most r neighbors $u \in V$ of v such that $j \in L_u$, then G admits a proper list-coloring from these lists.

Solution: Consider a random list-coloring c of G, where each $c(v)$ is selected from L_v independently and equiprobably. For an edge $e = \{u, v\} \in E$ and a color $j \in L_u \cap L_v$, let E_e^j be the event that $c(u) = c(v) = j$. The event E_e^j is independent of E_f^i when e and f are disjoint or when $j \notin L_{e \cap f}$, so E_e^j is only dependent of at most $d = 2 \cdot (r-1) \cdot 10r$ other events. Since

$$e(d+1) \Pr\left[E_e^j\right] = \frac{e(20r(r-1)+1)}{100r^2} < \frac{e}{5} < 1,$$

by the local lemma, $\Pr\left[\bigcap_{e,j} \overline{E_e^j}\right] > 0$, implying that there is a proper list-coloring of G from the given lists.

3. Let $k \geq 1$ be an integer, let G be a 2-connected graph, let x, y be distinct vertices of G, and assume that every vertex of G other than x or y has degree at least k. Prove that G has a path with ends x and y of length at least k.

Solution: We proceed by induction on k. The statement clearly holds for $k = 1$; thus we assume that $k \geq 2$ and that the statement holds for $k-1$. Let $G' := G \backslash x$. If G' is 2-connected, then let $x' \in V(G') - \{y\}$ be a neighbor of x. It exists, because G is 2-connected. Notice that every vertex of G' other than y has degree at least $k-1$. By induction there exists a path P' in G' from x' to y of length at least $k-1$; then $P' + x$ is as desired. Thus we may assume that G' is not 2-connected, and hence $G' = A \cup B$, where A and B are subgraphs of G' such that $|V(A) \cap V(B)| = 1$ and $V(A) - V(B) \neq \emptyset \neq V(B) - V(A)$. We may assume that $y \in V(B)$, and that A is minimal. It follows that A is 2-connected or isomorphic to K_2. Let y' be the unique vertex in $V(A) \cap V(B)$. Since G is 2-connected, x has a neighbor $x'' \in V(A) - \{y'\}$. By induction the graph A has a path P from x'' to y' of length at least $k-1$. Let Q be a path in B from y' to y. Then $P \cup Q + x$ is as desired.

4. Let $v_1, v_2, \dots v_n$ be n vectors from $\{\pm 1\}^n$ chosen uniformly and independently. Let M_n be the largest pairwise dot product in absolute value: i.e

$$M_n = \max_{i \neq j} |v_i \cdot v_j|$$

Prove that

$$\frac{M_n}{2\sqrt{n \ln n}} \to 1$$

in probability as $n \to \infty$.

Hint. Consider the first and second moment methods applied to the number of pairs of vectors whose dot product exceeds (and falls below, respectively) $2\sqrt{n \ln n}$.

Solution: To show $\frac{M_n}{2\sqrt{n \ln n}}$ converges to 1 in probability we must show that for every $\epsilon > 0$,

$$\Pr[\left|\frac{M_n}{2\sqrt{n \ln n}} - 1\right| > \epsilon] \to 0 \text{ as } n \to \infty$$

Therefore it is enough to show the following two facts:

(a) $\Pr[M_n \geq (1+\epsilon)2\sqrt{n \ln n}] \to 0$

(b) $\Pr[M_n \geq (1-\epsilon)2\sqrt{n \ln n}] \to 0$

To prove 1. we use the first-moment method. Let X be the number of pairs of vectors with dot product $\geq (1+\epsilon)2\sqrt{n \ln n}$. If $M_n \geq (1+\epsilon)2\sqrt{n \ln n}$, then $X \geq 1$. We will use Markov's Inequality, $\Pr[X \geq 1] \leq \mathbb{E}X$. We write

$$X = X_{1,2} + \dots + X_{i,j} + \dots$$

where $X_{i,j} = 1$ if $|v_i \cdot v_j| \geq (1+\epsilon)2\sqrt{n \ln n}$ and 0 otherwise.

$$\mathbb{E}X_{i,j} = \Pr[|v_i \cdot v_j| \geq (1+\epsilon)2\sqrt{n \ln n}] = \exp\left(-2(1+\epsilon)^2 \ln n(1+o(1))\right)$$

using a Chernoff bound, since $v_i \cdot v_j$ is distributed as a simple symmetric random walk of n steps, and so

$$\begin{aligned} \mathbb{E}X &= \binom{n}{2}\mathbb{E}X_{i,j} \\ &\leq \frac{n^2}{2}\frac{1}{n^{2(1+\epsilon)^2}} = o(1) \end{aligned}$$

which proves 1.

To prove 2. we use the second-moment method. Let Y be the number of pairs of vectors with dot product $\geq (1-\epsilon)2\sqrt{n \ln n}$. Similar to the above, we let $Y_{i,j} = 1$ if $|v_i \cdot v_j| \geq (1-\epsilon)2\sqrt{n \ln n}$ and 0 otherwise. Then we have

$$\begin{aligned} \mathbb{E}Y &= \binom{n}{2}\mathbb{E}Y_{i,j} \\ &\geq \frac{n^2}{2}\frac{1}{n^{2(1-\epsilon)^2}} = \omega(1) \end{aligned}$$

To bound the variance, we write

$$\begin{aligned} \mathrm{var}(Y) &= \sum_{i \neq j} \mathrm{var}(Y_{i,j}) + \sum_{(i,j) \neq (k,l)} \mathrm{cov}(Y_{i,j}, Y_{k,l}) \\ &\leq \mathbb{E}Y + \sum_{(i,j) \neq (k,l)} \mathrm{cov}(Y_{i,j}, Y_{k,l}) \end{aligned}$$

Now if (i,j) and (k,l) are disjoint pairs of pairs of vectors, then $Y_{i,j}$ and $Y_{k,l}$ are independent and so have covariance 0. If they overlap, say $Y_{i,j}$ and $Y_{i,k}$, the covariance is still 0: conditioned on $v_i \cdot v_j$, $v_i \cdot v_k$ still has the distribution of a SSRW of n steps. And so all the covariances are 0, giving $\mathrm{var}(Y) \leq \mathbb{E}(Y)$. Then we apply Chebyshev:

$$\Pr[Y = 0] \leq \frac{\mathrm{var}(Y)}{(\mathbb{E}Y)^2} \leq \frac{1}{\mathbb{E}Y} = o(1)$$

which completes the proof of 2.

5. Let G be a simple 3-regular graph, and let k be its edge-chromatic number. Prove that if every two k-edge-colorings of G differ by a permutation of colors, then $k = 3$ and G has three distinct Hamiltonian cycles.

Solution: By Vizing's theorem $k = 3$ or $k = 4$. Suppose for a contradiction that $k = 4$, and let $f : E(G) \to \{1, 2, 3, 4\}$ be a proper edge-coloring. Let H_{12} be the subgraph of G induced by edges e such that $f(e) \in \{1, 2\}$. Then H_{12} has maximum degree at most two, and it is a spanning subgraph, because every vertex is incident with an edge colored 1 or 2. Furthermore, H_{12} is connected, because otherwise swapping the colors 1 and 2 on one component of H_{12} would produce a k-edge-coloring that cannot be obtained from f by permuting colors. Thus H_{12} is a Hamilton path or Hamilton cycle. The same applies to the analogously defined graph H_{34}. However, H_{12} and H_{34} are edge-disjoint, and hence G has at most four vertices, contrary to the fact that $k = 4$.

Thus $k = 3$. Let us consider an arbitrary 3-edge-coloring of G. The union of every two color classes is a Hamiltonian cycle by the same argument as above. Thus G has three distinct Hamiltonian cycles, as required.

6. Show that there exists an absolute constant c so that if $\{S_i : 1 \le i \le n\}$ is any sequence of sets with $|S_i| \ge c$, for all $i = 1, 2, \ldots, n$, then there exists a sequence $\{x_i : 1 \le i \le n\}$ with $x_i \in S_i$, for all $i = 1, 2, \ldots, n$, which is square-free, i.e., there is no pair i, j with $1 \le i < j \le 2j - i - 1 \le n$ so that $x_{i+k} = x_{j+k}$ for all $k = 0, 1, \ldots, j - i - 1$. Hint: This is an application of the asymmetric version of the Lovasz Local Lemma.

Solution: Clearly, we may assume n is very large. To see, this simply expand the list of sets by adding arbitrary c elements sets. Any initial portion of a square-free string is square-free.

Now suppose that each set S_i has c elements (as usual c will be specified later). Then we form a word $x_1 x_2 x_3 \ldots x_n$ by making a random choice from each S_i with all elements of S_i being equally likely. For each pair (i, k) with $1 \le i < i + 2k - 1 \le n$, let $A(i, k)$ be the event that the length k substring $x_i x_{i+1} \ldots x_{i+k-1}$ is the first half of a square and is repeated in positions $x_{i+k} x_{i+k+1} \ldots x_{i+2k-1}$.

Since the characters in the string are chosen at random, we note that $\Pr[A(i, k)] \le 1/c^k$.

Clearly, the dependency neighborhood of $A(i, k)$ consists on those events $A(j, m)$ where $[i, i + 2k - 1] \cap [j, j + 2m - 1] \ne \emptyset$. So we group them according to the value of m. For each value of m, there are (at most) $2k + 2m - 1$ such events.

To apply the Local Lemma, we will set $x(i, k) = 1/d^k$ where d will be a constant depending on c and just a bit smaller. Now the inequality we need is:

$$\frac{1}{c^k} \leq \frac{1}{d^k} \prod_{m=1}^{n/2} (1 - \frac{1}{d^m})^{2k+2m-1}.$$

Multiplying both sides by d^k and taking logarithms, the preceding inequality becomes

$$k \ln(d/c) \leq \sum_{m=1}^{n/2} (2k + 2m - 1) \ln(1 - \frac{1}{d^m}).$$

Recall that when $|x| < 1$,

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$

Taking derivatives we have

$$\frac{1}{(1-x)^2} = \sum_{m=1}^{\infty} m x^{m-1}.$$

We use these two formulas, the approximation $\ln(1-1/d^m)$ by $-1/d^m$ and multiply both sides by -1, to obtain:

$$\begin{aligned} k \ln(c/d) &\geq \sum_{m=1}^{n/2} (2k + 2m - 1) \frac{1}{d^m} \\ &\sim \frac{2k-1}{d} \sum_{m=0}^{\infty} \frac{1}{d^m} + \frac{2}{d} \sum_{m=1}^{\infty} m \frac{1}{d^{m-1}} \\ &= \frac{2k-1}{d} \frac{d}{d-1} + \frac{2}{d} \frac{d^2}{(d-1)^2} \end{aligned}$$

Now it is easy to see that suitable choices for c and d can be found.