1. Prove that for every integer $k \geq 1$ there exists an integer $N$ such that if the subsets of $\{1, 2, \ldots, N\}$ are colored using $k$ colors, then there exist disjoint non-empty sets $X, Y \subseteq \{1, 2, \ldots, N\}$ such that $X, Y$ and $X \cup Y$ receive the same color.

*Hint.* You may want to consider intervals.

**Solution:** By Ramsey’s theorem there exists an integer $N$ such that for every $k$-coloring of 2-element subsets of $\{1, 2, \ldots, N+1\}$ there exists a 3-element set $A \subseteq \{1, 2, \ldots, N+1\}$ such that all 2-element subsets of $A$ receive the same color. We claim that $N$ satisfies the requirements of the problem. For $i, j \in \{1, 2, \ldots, N+1\}$ with $i < j$ we color the set $\{i, j\}$ using the color of the set $\{i, i+1, \ldots, j-1\}$. By the choice of $N$ there exist $i, j, k \in \{1, 2, \ldots, N+1\}$ such that $i < j < k$ and the sets $\{i, j\}, \{j, k\}$ and $\{i, k\}$ receive the same color. Then the sets $X := \{i, i+1, \ldots, j-1\}$ and $Y := \{j, j+1, \ldots, k-1\}$ are as desired.

2. A proper list-coloring of a graph $G = (V, E)$ from lists $\{L_v \subset \mathbb{N} \mid v \in V\}$ is a function $c : V \rightarrow \mathbb{N}$ such that $c(v) \in L_v$ for all $v \in V$ and $c(u) \neq c(v)$ for all $\{u, v\} \in E$.

Let $r$ be a natural number. Prove that if for all $v \in V$ we have $|L_v| = 10r$ and for all $j \in L_v$ there are at most $r$ neighbors $u \in V$ of $v$ such that $j \in L_u$, then $G$ admits a proper list-coloring from these lists.

**Solution:** Consider a random list-coloring $c$ of $G$, where each $c(v)$ is selected from $L_v$ independently and equiprobably. For an edge $e = \{u, v\} \in E$ and a color $j \in L_u \cap L_v$, let $E^j_e$ be the event that $c(u) = c(v) = j$. The event $E^j_e$ is independent of $E^i_f$ when $e$ and $f$ are disjoint or when $j \notin L_{e \cap f}$, so $E^j_e$ is only dependent of at most $d = 2 \cdot (r - 1) \cdot 10r$ other events. Since

$$e(d + 1) \Pr[E^j_e] = \frac{e(20r(r-1)+1)}{100r^2} < \frac{e}{5} < 1,$$

by the local lemma, $\Pr[\bigcap_{e,j} \overline{E^j_e}] > 0$, implying that there is a proper list-coloring of $G$ from the given lists.

3. Let $k \geq 1$ be an integer, let $G$ be a 2-connected graph, let $x, y$ be distinct vertices of $G$, and assume that every vertex of $G$ other than $x$ or $y$ has degree at least $k$. Prove that $G$ has a path with ends $x$ and $y$ of length at least $k$. 


Solution: We proceed by induction on $k$. The statement clearly holds for $k = 1$; thus we assume that $k \geq 2$ and that the statement holds for $k - 1$. Let $G' := G \setminus x$. If $G'$ is 2-connected, then let $x' \in V(G') - \{y\}$ be a neighbor of $x$. It exists, because $G$ is 2-connected. Notice that every vertex of $G'$ other than $y$ has degree at least $k - 1$. By induction there exists a path $P'$ in $G'$ from $x'$ to $y$ of length at least $k - 1$; then $P' + x$ is as desired. Thus we may assume that $G'$ is not 2-connected, and hence $G' = A \cup B$, where $A$ and $B$ are subgraphs of $G'$ such that $|V(A) \cap V(B)| = 1$ and $V(A) - V(B) \neq \emptyset \neq V(B) - V(A)$. We may assume that $y \in V(B)$, and that $A$ is minimal. It follows that $A$ is 2-connected or isomorphic to $K_2$. Let $y'$ be the unique vertex in $V(A) \cap V(B)$. Since $G$ is 2-connected, $x$ has a neighbor $x'' \in V(A) - \{y'\}$. By induction the graph $A$ has a path $P$ from $x''$ to $y'$ of length at least $k - 1$. Let $Q$ be a path in $B$ from $y'$ to $y$. Then $P \cup Q + x$ is as desired.

4. Let $v_1, v_2, \ldots, v_n$ be $n$ vectors from $\{\pm 1\}^n$ chosen uniformly and independently. Let $M_n$ be the largest pairwise dot product in absolute value: i.e

$$M_n = \max_{i \neq j} |v_i \cdot v_j|$$

Prove that

$$\frac{M_n}{2\sqrt{n \ln n}} \to 1$$

in probability as $n \to \infty$.

**Hint.** Consider the first and second moment methods applied to the number of pairs of vectors whose dot product exceeds (and falls below, respectively) $2\sqrt{n \ln n}$.

**Solution:** To show $\frac{M_n}{2\sqrt{n \ln n}}$ converges to 1 in probability we must show that for every $\epsilon > 0$,

$$\Pr\left[\left\|\frac{M_n}{2\sqrt{n \ln n}} - 1\right\| > \epsilon\right] \to 0 \text{ as } n \to \infty$$

Therefore it is enough to show the following two facts:

(a) $\Pr[M_n \geq (1 + \epsilon)2\sqrt{n \ln n}] \to 0$

(b) $\Pr[M_n \geq (1 - \epsilon)2\sqrt{n \ln n}] \to 0$

To prove 1. we use the first-moment method. Let $X$ be the number of pairs of vectors with dot product $\geq (1 + \epsilon)2\sqrt{n \ln n}$. If $M_n \geq (1 + \epsilon)2\sqrt{n \ln n}$, then $X \geq 1$. We will use Markov’s Inequality, $\Pr[X \geq 1] \leq E[X]$. We write

$$X = X_{1,2} + \cdots + X_{i,j} + \cdots$$
where $X_{i,j} = 1$ if $|v_i \cdot v_j| \geq (1 + \epsilon)2\sqrt{n \ln n}$ and 0 otherwise.

$$\mathbb{E}X_{i,j} = \Pr[|v_i \cdot v_j| \geq (1 + \epsilon)2\sqrt{n \ln n}] = \exp \left( -2(1 + \epsilon)^2 \ln n \right) = o(1)$$

using a Chernoff bound, since $v_i \cdot v_j$ is distributed as a simple symmetric random walk of $n$ steps, and so

$$\mathbb{E}X = \left( \frac{n}{2} \right) \mathbb{E}X_{i,j}$$

$$\leq \frac{n^2}{2} \frac{1}{n^{2(1+\epsilon)^2}} = o(1)$$

which proves 1.

To prove 2, we use the second-moment method. Let $Y$ be the number of pairs of vectors with dot product $\geq (1 - \epsilon)2\sqrt{n \ln n}$. Similar to the above, we let $Y_{i,j} = 1$ if $|v_i \cdot v_j| \geq (1 - \epsilon)2\sqrt{n \ln n}$ and 0 otherwise. Then we have

$$\mathbb{E}Y = \left( \frac{n}{2} \right) \mathbb{E}Y_{i,j}$$

$$\geq \frac{n^2}{2} \frac{1}{n^{2(1-\epsilon)^2}} = \omega(1)$$

To bound the variance, we write

$$\text{var}(Y) = \sum_{i \neq j} \text{var}(Y_{i,j}) + \sum_{(i,j) \neq (k,l)} \text{cov}(Y_{i,j}, Y_{k,l})$$

$$\leq \mathbb{E}Y + \sum_{(i,j) \neq (k,l)} \text{cov}(Y_{i,j}, Y_{k,l})$$

Now if $(i, j)$ and $(k, l)$ are disjoint pairs of pairs of vectors, then $Y_{i,j}$ and $Y_{k,l}$ are independent and so have covariance 0. If they overlap, say $Y_{i,j}$ and $Y_{i,k}$, the covariance is still 0: conditioned on $v_i \cdot v_j$, $v_i \cdot v_k$ still has the distribution of a SSRW of $n$ steps. And so all the covariances are 0, giving $\text{var}(Y) \leq \mathbb{E}(Y)$. Then we apply Chebyshev:

$$\Pr[Y = 0] \leq \frac{\text{var}(Y)}{\mathbb{E}Y^2} \leq \frac{1}{\mathbb{E}Y} = o(1)$$

which completes the proof of 2.
5. Let $G$ be a simple 3-regular graph, and let $k$ be its edge-chromatic number. Prove that if every two $k$-edge-colorings of $G$ differ by a permutation of colors, then $k = 3$ and $G$ has three distinct Hamiltonian cycles.

**Solution:** By Vizing's theorem $k = 3$ or $k = 4$. Suppose for a contradiction that $k = 4$, and let $f : E(G) \to \{1, 2, 3, 4\}$ be a proper edge-coloring. Let $H_{12}$ be the subgraph of $G$ induced by edges $e$ such that $f(e) \in \{1, 2\}$. Then $H_{12}$ has maximum degree at most two, and it is a spanning subgraph, because every vertex is incident with an edge colored 1 or 2. Furthermore, $H_{12}$ is connected, because otherwise swapping the colors 1 and 2 on one component of $H_{12}$ would produce a $k$-edge-coloring that cannot be obtained from $f$ by permuting colors. Thus $H_{12}$ is a Hamilton path or Hamilton cycle. The same applies to the analogously defined graph $H_{34}$. However, $H_{12}$ and $H_{34}$ are edge-disjoint, and hence $G$ has at most four vertices, contrary to the fact that $k = 4$.

Thus $k = 3$. Let us consider an arbitrary 3-edge-coloring of $G$. The union of every two color classes is a Hamiltonian cycle by the same argument as above. Thus $G$ has three distinct Hamiltonian cycles, as required.

6. Show that there exists an absolute constant $c$ so that if $\{S_i : 1 \leq i \leq n\}$ is any sequence of sets with $|S_i| \geq c$, for all $i = 1, 2, \ldots, n$, then there exists a sequence $\{x_i : 1 \leq i \leq n\}$ with $x_i \in S_i$, for all $i = 1, 2, \ldots, n$, which is square-free, i.e., there is no pair $i, j$ with $1 \leq i < j \leq 2j - i - 1 \leq n$ so that $x_{i+k} = x_{j+k}$ for all $k = 0, 1, \ldots, j - i - 1$. Hint: This is an application of the asymmetric version of the Lovasz Local Lemma.

**Solution:** Clearly, we may assume $n$ is very large. To see, this simply expand the list of sets by adding arbitrary $c$ elements sets. Any initial portion of a square-free string is square-free.

Now suppose that each set $S_i$ has $c$ elements (as usual $c$ will be specified later). Then we form a word $x_1x_2x_3 \ldots x_n$ by making a random choice from each $S_i$ with all elements of $S_i$ being equally likely. For each pair $(i, k)$ with $1 \leq i < i+2k-1 \leq n$, let $A(i, k)$ be the event that the length $k$ substring $x_ix_{i+1} \ldots x_{i+k-1}$ is the first half of a square and is repeated in positions $x_{i+k}x_{i+k+1} \ldots x_{i+2k-1}$.

Since the characters in the string are chosen at random, we note that $\Pr[A(i, k)] \leq 1/c^k$.

Clearly, the dependency neighborhood of $A(i, k)$ consists on those events $A(j, m)$ where $[i, i+2k-1] \cap [j, j+2m-1] \neq \emptyset$. So we group them according to the value of $m$. For each value of $m$, there are (at most) $2k + 2m - 1$ such events.

To apply the Local Lemma, we will set $x(i, k) = 1/d^k$ where $d$ will be a constant.
depending on $c$ and just a bit smaller. Now the inequality we need is:

$$\frac{1}{c^k} \leq \frac{1}{d^k} \prod_{m=1}^{n/2} (1 - \frac{1}{d^m})^{2k+2m-1}.$$ 

Multiplying both sides by $d^k$ and taking logarithms, the preceding inequality becomes

$$k \ln(d/c) \leq \sum_{m=1}^{n/2} (2k + 2m - 1) \ln(1 - \frac{1}{d^m}).$$

Recall that when $|x| < 1$,

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$ 

Taking derivatives we have

$$\frac{1}{(1-x)^2} = \sum_{m=1}^{\infty} mx^{m-1}.$$ 

We use these two formulas, the approximation $\ln(1 - 1/d^m)$ by $-1/d^m$ and multiply both sides by $-1$, to obtain:

$$k \ln(c/d) \geq \sum_{m=1}^{n/2} (2k + 2m - 1) \frac{1}{d^m}$$

$$\sim \frac{2k-1}{d} \sum_{m=0}^{\infty} \frac{1}{d^m} + \frac{2}{d} \sum_{m=1}^{\infty} m \frac{1}{d^m-1}$$

$$= \frac{2k-1}{d(d-1)} + \frac{2}{d(d-1)^2}$$

Now it is easy to see that suitable choices for $c$ and $d$ can be found.