New Algebra Comprehensive Exam Questions

1. Let F be an algebraicaly closed field, V a finite-dimensional vector space over F, and $T : V \to V$ a linear transformation. Make V into a module over the polynomial ring $F[x]$ by requiring that $x \cdot v = T(v)$. Prove that if T has distinct eigenvalues, then V is a cyclic $F[x]$-module. Is the converse true? Give a proof or counterexample.

Solution: Since T has distinct eigenvlaues, there is a basis $v_1, \ldots, v_n$ of V consisting of eigenvectors of T, say $T(v_i) = \lambda_i v_i$ where $\lambda_i \in F$ and the λ_i are distinct. Let

$$v = \sum_{i=1}^{n} v_i.$$

I claim that the $F[x]$-submodule of V generated by v is all of V, so V is cyclic. To see this, let $f_i = \prod_{k \neq i}(x - \lambda_k)$. Then we have

$$f_i v_j = \begin{cases} 0 & \text{if } i \neq j \\ \prod_{k \neq i}(\lambda_i - \lambda_k) v_i & \text{if } i = j. \end{cases}$$

Thus $f_i v$ is a non-zero multiple of v_i. This shows that for all i, v_i is in the submodule of V generated by v. Since the v_i form a basis of V, this submodule is all of V.

To see the converse is false, take $V = F^2$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then V is cyclic (generated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) but T does not have distinct eigenvalues.

2. Let R be a commutative ring and let $I \subset R$ and $J \subset R$ be ideals. Consider the map $\phi : I \otimes_R J \to IJ$ which sends $a \otimes b$ to ab. Prove or give a counterexample to the statements: "ϕ is onto" and "ϕ is one-to-one."

Solution: By definition, IJ is the set of finite sums of products $\sum_{i=1}^{n} a_i b_i$ where the $a_i \in I$ and the $b_i \in J$. Such a sum is $\phi(\sum_{i=1}^{n} a_i \otimes b_i)$, so ϕ is onto.

On the other hand, ϕ is not one-to-one in general. For example, take R to be the polynomial ring $k[x, y]$ over a field k and let $I = J = (x, y)$. Then $x \otimes y - y \otimes x$ obviously maps to zero in I^2, but it is not zero in $I \otimes_R I$. (To see this, we need to produce an R-module M and a bilinear map $\psi : I \times I \to M$ such that $\psi(x, y) \neq \psi(y, x)$. Let $M = k$ with x and y acting as multiplication by 0 and note that to specifiy an R-module homomorphism $I \to M$ we may assign arbitrary values to x and y. Define two such homomorphisms by $\psi_1(x) = \psi_2(y) = 1$ and $\psi_2(y) = \psi_1(x) =$. Then $\psi(f, g) = \psi_1(f)\psi_2(g)$ defines a bilinear map $I \times I \to M$, and we have $\psi(x, y) = 1$ and $\psi(y, x) = 0$ as desired.)

3. Let P be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$. Exhibit an exact sequence of $\mathbb{Z}$-modules

$$0 \to M \to N \to N/M \to 0$$

such that

$$0 \to \mathrm{Hom}(N/M, P) \to \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P) \to 0$$

is not exact.

Solution: Let $M = N = \mathbb{Z}$ and let $M \to N$ be multiplication by 2, so that our sequence is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

We have $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{2} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

is the zero map, so the Hom-sequence fails to be exact on the right.

4. Find the Galois group of the splitting field for $f(x) = x^3 - 7$ over $K = \mathbf{Q}(\sqrt{-3})$.

Solution: First of all, note that f is irreducible over $\mathbf{Q}$, for example by Eisenstein's criterion. Next, since $[K : \mathbf{Q}] = 2$ and $[\mathbf{Q}(\sqrt[3]{7}) : \mathbf{Q}] = 3$ are relatively prime, we know that $[\mathbf{Q}(\sqrt[3]{7}, \sqrt{-3}) : \mathbf{Q}] = 6$ and thus $[\mathbf{Q}(\sqrt[3]{7}, \sqrt{-3}) : K] = 3$. As K contains a primitive cube root of unity $\omega = \frac{1+\sqrt{-3}}{2}$, f splits into distinct linear factors over K:

$$f(x) = (x - \sqrt[3]{7})(x - \omega\sqrt[3]{7})(x - \omega^2\sqrt[3]{7}).$$

It follows that $\mathbf{Q}(\sqrt[3]{7})$ is a splitting field for f over K with Galois group isomorphic to $\mathbf{Z}/3\mathbf{Z}$.

5. Let ζ be a primitive 37th root of unity, and let $\eta = \zeta + \zeta^{10} + \zeta^{26}$. Determine the Galois group of $\mathbf{Q}(\eta)$ over $\mathbf{Q}$.

Solution: It is a standard fact from Galois theory that $L = \mathbf{Q}(\zeta)$ is Galois over $\mathbf{Q}$ with Galois group G isomorphic to the cyclic group $(\mathbf{Z}/37\mathbf{Z})^*$ of order 36. Since $K = \mathbf{Q}(\eta)$ is a subfield of L, its Galois group H over $\mathbf{Q}$ is a quotient of G and hence is cyclic of degree $[K : \mathbf{Q}]$. It remains to determine this degree. The subset $\{1, 10, 26\} \subset (\mathbf{Z}/37\mathbf{Z})^*$ is in fact a subgroup. It follows that if $\sigma \in G$ is the map taking ζ to ζ^{10}, then η is fixed by the action of $H = \langle\sigma\rangle$, which has order 3. This implies that $K = \mathbf{Q}(\eta)$ is the fixed field of H in L. By Galois theory, we have $[L : K] = 3$ and therefore $[K : \mathbf{Q}] = 12$.