1. Let $F$ be an algebraically closed field, $V$ a finite-dimensional vector space over $F$, and $T: V \to V$ a linear transformation. Make $V$ into a module over the polynomial ring $F[x]$ by requiring that $x \cdot v = T(v)$. Prove that if $T$ has distinct eigenvalues, then $V$ is a cyclic $F[x]$-module. Is the converse true? Give a proof or counterexample.

**Solution:** Since $T$ has distinct eigenvalues, there is a basis $v_1, \ldots, v_n$ of $V$ consisting of eigenvectors of $T$, say $T(v_i) = \lambda_i v_i$ where $\lambda_i \in F$ and the $\lambda_i$ are distinct. Let

$$v = \sum_{i=1}^n v_i.$$ 

I claim that the $F[x]$-submodule of $V$ generated by $v$ is all of $V$, so $V$ is cyclic. To see this, let $f_i = \prod_{k \neq i} (x - \lambda_k)$. Then we have

$$f_i v_j = \begin{cases} 0 & \text{if } i \neq j \\ \prod_{k \neq i} (\lambda_i - \lambda_k) v_i & \text{if } i = j. \end{cases}$$

Thus $f_i v_j$ is a non-zero multiple of $v_i$. This shows that for all $i$, $v_i$ is in the submodule of $V$ generated by $v$. Since the $v_i$ form a basis of $V$, this submodule is all of $V$.

To see the converse is false, take $V = F^2$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $V$ is cyclic (generated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) but $T$ does not have distinct eigenvalues.

2. Let $R$ be a commutative ring and let $I \subset R$ and $J \subset R$ be ideals. Consider the map $\phi: I \otimes_R J \to IJ$ which sends $a \otimes b$ to $ab$. Prove or give a counterexample to the statements: “$\phi$ is onto” and “$\phi$ is one-to-one.”

**Solution:** By definition, $IJ$ is the set of finite sums of products $\sum_{i=1}^n a_i b_i$ where the $a_i \in I$ and the $b_i \in J$. Such a sum is $\phi(\sum_{i=1}^n a_i \otimes b_i)$, so $\phi$ is onto.

On the other hand, $\phi$ is not one-to-one in general. For example, take $R$ to be the polynomial ring $k[x, y]$ over a field $k$ and let $I = J = (x, y)$. Then $x \otimes y - y \otimes x$ obviously maps to zero in $I^2$, but it is not zero in $I \otimes_R I$. (To see this, we need to produce an $R$-module $M$ and a bilinear map $\psi: I \times I \to M$ such that $\psi(x, y) \neq \psi(y, x)$. Let $M = k$ with $x$ and $y$ acting as multiplication by 0 and note that to specify an $R$-module homomorphism $I \to M$ we may assign arbitrary values to $x$ and $y$. Define two such homomorphisms by $\psi_1(x) = \psi_2(y) = 1$ and $\psi_2(y) = \psi_1(x) = 0$. Then $\psi(f, g) = \psi_1(f)\psi_2(g)$ defines a bilinear map $I \times I \to M$, and we have $\psi(x, y) = 1$ and $\psi(y, x) = 0$ as desired.)
3. Let \( P \) be the \( \mathbb{Z} \)-module \( \mathbb{Z}/2\mathbb{Z} \). Exhibit an exact sequence of \( \mathbb{Z} \)-modules
\[
0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0
\]
such that
\[
0 \rightarrow \text{Hom}(N/M, P) \rightarrow \text{Hom}(N, P) \rightarrow \text{Hom}(M, P) \rightarrow 0
\]
is not exact.

**Solution:** Let \( M = N = \mathbb{Z} \) and let \( M \rightarrow N \) be multiplication by 2, so that our sequence is
\[
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.
\]
We have \( \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \) and
\[
\text{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{2} \text{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})
\]
is the zero map, so the Hom-sequence fails to be exact on the right.

4. Find the Galois group of the splitting field for \( f(x) = x^3 - 7 \) over \( K = \mathbb{Q}(\sqrt{-3}) \).

**Solution:** First of all, note that \( f \) is irreducible over \( \mathbb{Q} \), for example by Eisenstein’s criterion. Next, since \( [K : \mathbb{Q}] = 2 \) and \( [\mathbb{Q}(\sqrt{7}, \sqrt{-3}) : \mathbb{Q}] = 3 \) are relatively prime, we know that \( [\mathbb{Q}(\sqrt[3]{7}, \sqrt{-3}) : \mathbb{Q}] = 6 \) and thus \( [\mathbb{Q}(\sqrt[3]{7}, \sqrt{-3}) : K] = 3 \). As \( K \) contains a primitive cube root of unity \( \omega = \frac{1 + \sqrt{-3}}{2} \), \( f \) splits into distinct linear factors over \( K \):
\[
f(x) = (x - \sqrt[3]{7})(x - \omega \sqrt[3]{7})(x - \omega^2 \sqrt[3]{7}).
\]
It follows that \( \mathbb{Q}(\sqrt[3]{7}) \) is a splitting field for \( f \) over \( K \) with Galois group isomorphic to \( \mathbb{Z}/3\mathbb{Z} \).

5. Let \( \zeta \) be a primitive 37th root of unity, and let \( \eta = \zeta + \zeta^{10} + \zeta^{26} \). Determine the Galois group of \( \mathbb{Q}(\eta) \) over \( \mathbb{Q} \).

**Solution:** It is a standard fact from Galois theory that \( L = \mathbb{Q}(\zeta) \) is Galois over \( \mathbb{Q} \) with Galois group \( G \) isomorphic to the cyclic group \( (\mathbb{Z}/37\mathbb{Z})^* \) of order 36. Since \( K = \mathbb{Q}(\eta) \) is a subfield of \( L \), its Galois group \( H \) over \( \mathbb{Q} \) is a quotient of \( G \) and hence is cyclic of degree \( [K : \mathbb{Q}] \). It remains to determine this degree. The subset \( \{1, 10, 26\} \subset (\mathbb{Z}/37\mathbb{Z})^* \) is in fact a subgroup. It follows that if \( \sigma \in G \) is the map taking \( \zeta \) to \( \zeta^{10} \), then \( \eta \) is fixed by the action of \( H = \langle \sigma \rangle \), which has order 3. This implies that \( K = \mathbb{Q}(\eta) \) is the fixed field of \( H \) in \( L \). By Galois theory, we have \( [L : K] = 3 \) and therefore \( [K : \mathbb{Q}] = 12 \).