1. Assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and \(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\) are subfields of \(\mathcal{F}\) such that \(\mathcal{G}_1 \vee \mathcal{G}_2\) is independent from \(\mathcal{G}_3\). Assume that \(X\) is a \(\mathcal{G}_1\)-measurable and integrable random variable. Show that \(\mathbb{E}[X|\mathcal{G}_2 \vee \mathcal{G}_3] = \mathbb{E}[X|\mathcal{G}_2]\). Here \(\mathcal{G}_1 \vee \mathcal{G}_2\) is the smallest sigma algebra containing both \(\mathcal{G}_1\) and \(\mathcal{G}_2\).

**Solution:** From the definition of conditional expectation, for some \(\mathcal{G}\)-measurable r.v. \(Y\) we have
\[
\mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[Y \mathbf{1}_A] \quad \forall A \in \mathcal{G}
\]
then \(Y = \mathbb{E}[X|\mathcal{G}]\).

Let \(Y = \mathbb{E}[X|\mathcal{G}_2]\). To prove the claim we need to show
\[
\mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[Y \mathbf{1}_A] \quad \forall A \in \mathcal{G}_2 \vee \mathcal{G}_3.
\]
Define \(\Lambda = \{A \in \mathcal{G}_2 \vee \mathcal{G}_3 : \mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[Y \mathbf{1}_A]\}\). From this definition it follows that \(\Lambda\) is a \(\lambda\)-system (this follows from linearity of expectation). Now consider the set \(\Pi = \{G \cap H : G \in \mathcal{G}_2, H \in \mathcal{G}_3\}\). Clearly \(\Pi \subset \Lambda\) and \(\sigma(\Pi) = \mathcal{G}_2 \vee \mathcal{G}_3\). Then by the \(\pi - \lambda\) theorem, to prove the claim it is sufficient to show that \(\mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[Y \mathbf{1}_A] \forall A \in \Pi\).

Clearly for any \(G \in \mathcal{G}_2, H \in \mathcal{G}_3\) we have \(\mathbf{1}_H\) is independent of \(X \mathbf{1}_G\) and \(Y \mathbf{1}_G\) therefore
\[
\mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[X \mathbf{1}_{G \cap H}] = \mathbb{E}[X \mathbf{1}_G \mathbf{1}_H] = \mathbb{E}[X \mathbf{1}_G] \mathbb{E}[\mathbf{1}_H]
\]
and
\[
\mathbb{E}[Y \mathbf{1}_A] = \mathbb{E}[Y \mathbf{1}_{G \cap H}] = \mathbb{E}[Y \mathbf{1}_G \mathbf{1}_H] = \mathbb{E}[Y \mathbf{1}_G] \mathbb{E}[\mathbf{1}_H]
\]
Also, \(\mathbb{E}[X \mathbf{1}_G] = \mathbb{E}[Y \mathbf{1}_G]\), because \(Y = \mathbb{E}[X|\mathcal{G}_2]\) and \(G \in \mathcal{G}_2\). therefore
\[
\mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[Y \mathbf{1}_A]
\]
Since this holds for an arbitrary \(A \in \Pi\), it holds for all \(\Pi\), and we are done. We conclude
\[
\mathbb{E}[X|\mathcal{G}_2 \vee \mathcal{G}_3] = Y = \mathbb{E}[X|\mathcal{G}_2].
\]

2. Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, \((\mathcal{F}_n)_{n \geq 0}\) is a filtration, and \((A_n)_{n \geq 0}\) is a non-decreasing sequence of random variables such that
   a) \(A_0 = 0\)
   b) \(A_n\) is \(\mathcal{F}_n\) measurable
   c) \(\mathbb{E}[A_n^2]\) is finite.

Also assume that \((B_n)_{n \geq 0}\) is a sequence random variables such that
   i) \(0 < \mathbb{E}[B_n^2] < \infty\) and \(\mathbb{E}[B_n] = 0\) for any \(n \geq 0\)
ii) $B_n$ is $\mathcal{F}_n$ measurable 

iii) $B_n$ is independent of $\mathcal{F}_{n-1}$ for each $n \geq 1$.

(a) Show that if $(M_n)_{n \geq 0}$ is a square integrable martingale such that $M_0 = 0$ and $(M_n^2 + A_n)_{n \geq 0}$ is a supermartingale, then $M_n = A_n = 0$ almost surely for any $n \geq 0$.

(b) If $A_n$ is $\mathcal{F}_{n-1}$ measurable for each $n \geq 1$, find a martingale $(M_n)_{n \geq 0}$ such that $(M_n^2 - A_n)_{n \geq 0}$ is a martingale.

**Solution:**

(a) Since $(M_n)_{n \geq 0}$ is a martingale, we know have

\[
\mathbb{E}[M_{n+1}^2 | \mathcal{F}_n] = \mathbb{E}[(M_{n+1} - M_n)^2 | \mathcal{F}_n] + 2M_n \mathbb{E}[(M_{n+1} - M_n) | \mathcal{F}_n] + \mathbb{E}[M_n^2 | \mathcal{F}_n] \\
= M_n^2 + \mathbb{E}[(M_{n+1} - M_n)^2 | \mathcal{F}_n]
\]

Since, $M_n^2 + a_n$ is a supermartingale, we get that

\[
\mathbb{E}[M_{n+1}^2 + A_{n+1} | \mathcal{F}_n] \leq M_n^2 + A_n
\]

and thus combining this with the above we obtain that

\[
\mathbb{E}[A_{n+1} - A_n | \mathcal{F}_n] + \mathbb{E}[(M_{n+1} - M_n)^2 | \mathcal{F}_n] \leq 0.
\]

Integrating this yields that

\[
\mathbb{E}[A_{n+1} - A_n] + \mathbb{E}[(M_{n+1} - M_n)^2] \leq 0.
\]

Since $A_{n+1} \geq A_n$ we conclude that almost surely, $M_{n+1} = M_n$ and $A_{n+1} = A_n$.

Induction finishes the proof.

(b) For the second part, we can construct the martingale in the following form:

\[
M_n = \sum_{k=1}^{n} B_k C_k
\]

where $C_k$ we choose to be $\mathcal{F}_{k-1}$ measurable. The martingale condition is then automatically satisfied because $B_k$ is independent of $\mathcal{F}_{k-1}$ and has mean 0. In order to satisfy the second property, notice that

\[
\mathbb{E}[M_n^2 - A_n | \mathcal{F}_{n-1}] = M_{n-1}^2 - A_n + \mathbb{E}[(M_n - M_{n-1})^2 | \mathcal{F}_{n-1}] = M_{n-1}^2 - A_n + \mathbb{E}[B_n^2 C_n^2 | \mathcal{F}_{n-1}] \\
= M_{n-1}^2 - A_n + C_n^2 \mathbb{E}[B_n^2].
\]

Thus, if we want this to be equal to $M_{n-1}^2 - A_{n-1}$, then we need to choose $C_n$ such that

\[
C_n^2 \mathbb{E}[B_n^2] = A_n - A_{n-1}
\]

which is possible with

\[
C_n = \sqrt{\frac{A_n - A_{n-1}}{\mathbb{E}[B_n^2]}}.
\]
3. Assume that $X(t)$ is a simple Poisson process. Find the joint distribution of $(X(t_1), X(t_2))$ and then the conditional expectation $E[X(t_1)|X(t_2)]$.

**Solution:** We have to distinguish two cases, one is $t_1 \geq t_2$ and $t_2 > t_1$.

In the case $t_1 \geq t_2$, it is easy to see from independence of increments that

$$E[X(t_1)|X(t_2)] = E[X_{t_1} - X_{t_2}|X(t_2)] + X_{t_2} = X_{t_2} + (t_1 - t_2).$$

For the case of $t_1 < t_2$ we need the joint distribution, which we can find using the fact that

$$P(X(t_2) = k + n, X(t_1) = n) = \frac{e^{-(t_2-t_1)}((t_2-t_1))^k e^{-t_1} t_1^n}{n!} k, n \in \mathbb{N}.$$ 

Therefore the conditional distribution of $(X(t_1)|X(t_2))$ is

$$P(X(t_1) = n|X(t_2) = k + n) = \frac{e^{-(t_2-t_1)}((t_2-t_1))^k e^{-t_1} t_1^n}{(k+n)!} \left(\frac{k!}{n!}\right) \left(1 - \frac{t_1}{t_2}\right)^k k, n \in \mathbb{N}.$$ 

So the distribution of $X(t_1)$ conditioned on $X(t_2)$ is a Binomial distribution with parameters $(X(t_2), \frac{t_1}{t_2})$, which then implies

$$E[X(t_1)|X(t_2)] = \frac{t_1}{t_2} X(t_2).$$

4. (a) Let $X$ be a real-valued r.v. on a probability space $\Omega, \mathcal{F}, P$ with density $f(x) = \frac{1}{3} \mathbb{I}_{[0,3]}(x)$. Find the correct assertions.

i. $P(X \in (0, 3)) = 1$.

ii. For all $\omega \in \Omega$, $X(\omega) \in (0, 3)$.

iii. For all $\omega \in \Omega$, $X(\omega) \in [0, 3]$.

(b) Let $(X_n)_n$ be a sequence of real-valued random variables. Find and justify the correct assertions.

i. $\{\sup_{n \geq 1} X_n < \infty\}$ is an asymptotic event.

ii. $\{\sup_{n \geq 1} X_n < c\}$ for some $c \in \mathbb{R}$ is an asymptotic event.

**Solution:** (a) $P(X \in (0, 3)) = 1$; (b) $\{\sup_{n \geq 1} X_n < \infty\}$ is an asymptotic event.

5. Let $X$ be a r.v. with Cauchy distribution $\mathcal{C}(1)$ (that means with density $f(x) = \frac{1}{\pi(1+x^2)}$ w.r.t. the Lebesgue measure on $\mathbb{R}$).
a) Determine the density of \( Z = X^{-1} \).

b) Determine the density \( f \) of \( \log |X| \).

**Solution:** (a) \( Z \) also follows Cauchy distribution \( C(1) \). (b) \( Z = \log|X| \) admits p.d.f. 
\[
f(z) = \frac{1}{\pi \cosh(z)}.
\]

6. Let \((X_n)_n\) be a sequence of real-valued random variables with respective densities \( f_n(x) = \frac{n^2}{2} e^{-n^2|x|} \).

(a) Compute for all \( n \in \mathbb{N}^* \) \( \mathbb{P}(|X_n| > n^{-3/2}) \)
(b) Compute \( \mathbb{P}(\limsup \{|X_n| > n^{-3/2}\}) \)
(c) What is the probability that \( \sum_n X_n \) converges absolutely?

**Solution:** (a) We have \( \mathbb{P} \left( \{|X_n| > n^{-3/2}\} \right) = \int_{|x| > n^{-3/2}} f_n(x) \, dx = e^{-\sqrt{n}}. \) (b) Since 
\[
\sum_{n \geq 1} e^{-\sqrt{n}} < +\infty,
\]
Borel-Cantelli’s Lemma gives \( \mathbb{P} \left( \limsup \{|X_n| > n^{-3/2}\} \right) = 0. \) (c) Set \( A = \limsup \{|X_n| > n^{-3/2}\}. \) Therefore the series \( \sum_{n \geq 1} X_n(w) \) converges absolutely for any \( w \in A^c \) and \( A^c \) is of probability 1.

7. Let \((X_n)_n\) be a sequence of independent and identically distributed real-valued random variables. Show that \( \frac{X_n}{n} \) converges almost surely to 0 if and only if \( X_1 \) is integrable.

**Solution:** Recall that \( X_n/n \) converges a.s. to 0 if and only if for any \( \varepsilon > 0 \), we have \( \mathbb{P}(\limsup \{|X_n|/n > \varepsilon\}) = 0. \) Since the \( X_n \) are independent, this is equivalent to 
\[
\sum_{n \geq 1} \mathbb{P} \left( \{|X_n|/n > \varepsilon\} \right) < \infty
\]
for any \( \varepsilon > 0. \) Next, since the \( X_n \) are identically distributed, we have \( \mathbb{P} \left( \{|X_1|/n > \varepsilon\} \right) = \mathbb{P} \left( \{|X_1|/\varepsilon| > t\} \right) dt \) for any \( \varepsilon > 0 \) and \( \varepsilon > 0. \) The previous condition is equivalent to 
\[
\sum_{n \geq 1} \mathbb{P} \left( \{|X_1|/\varepsilon| > t\} \right) < \infty
\]
for any \( \varepsilon > 0 \) since \( \mathbb{E}[|X_1|/\varepsilon] = \int_0^\infty \mathbb{P}(\{|X_1|/\varepsilon > t\}) \, dt. \)

8. Let \((X_n)_{n \geq 1}\) be a sequence of i.i.d random variables with standard Gaussian distribution \( N(0, 1) \). We recall that \( \mathbb{E}[e^{X_1}] = e^{1/2}. \) For all \( n \geq 1, \) set \( S_n = \sum_{i=1}^n X_i \) and \( M_n = e^{S_n - \frac{n}{2}}. \)
(a) Justify the a.s. convergence of $S_n$ and determine the limit.
(b) Show that $M_n \to 0$ a.s. as $n \to +\infty$.
(c) For any $n \geq 1$, compute $\mathbb{E}[M_n]$.
(d) Do we have $M_n \to 0$ in $L_1$? Justify your answer.
(e) Let $(a_n)_{n\geq1}$ be a sequence of real numbers such that $\sum_n a_n^2 < \infty$. Show that $\sum_{n\geq1} a_n X_n$ converges to a random variable a.s. and in $L^2$.

**Solution:** (a) $(X_n)_n$ is an i.i.d. sequence of integrable random variables. Thus the strong law of large numbers gives $\frac{S_n}{n} \to \mathbb{E}[X_1] = 0$ a.s. (b) This is a straightforward consequence of (a) since $M_n = e^{n(\frac{S_n}{n} - 1/2)}$. (c) Since the sequence $(X_n)_n$ is i.i.d., we have $\mathbb{E}[M_n] = (\mathbb{E}[e^{X_1}])^n e^{-\frac{n}{2}} = 1$. (d) We proceed by contradiction. Assume that $M_n$ converges in $L_1$ to a random variable $M$. On the one hand, this implies that $\mathbb{E}[M_n] \to \mathbb{E}[M]$ as $n \to \infty$. In view of (c), we then have $\mathbb{E}[M] = 1$. On the other hand, there exists a subsequence of $M_n$ that converges almost surely to $M$. Since $M_n$ converges almost surely to 0, this implies that $M = 0$ a.s. This contradicts that $\mathbb{E}[M] = 1$. Therefore $M_n$ does not converge in $L_1$. (e) We have for any integer $N \geq 1$ that $\sum_{n=1}^{N} a_n X_n \sim N(0, \sum_{k=1}^{N} a_k^2)$. By assumption the series $\sum_n a_n^2$ converges to $\sigma^2 = \sum_{n\geq1} a_n^2$. Thus, $\sum_{n=1}^{N} a_n X_n$ converges in distribution to a random variable $Z \sim N(0, \sigma^2)$. Levy’s theorem guarantees that $\sum_{n=1}^{N} a_n X_n$ also converges to $Z$ a.s. and in $L^2$. 