Probability Comprehensive Exam Questions

1. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are subfields of $\mathcal{F}$ such that $\mathcal{G}_1 \vee \mathcal{G}_2$ is independent from $\mathcal{G}_3$. Assume that X is a $\mathcal{G}_1$-measurable and integrable random variable. Show that $\mathbb{E}[X|\mathcal{G}_2 \vee \mathcal{G}_3] = \mathbb{E}[X|\mathcal{G}_2]$. Here $\mathcal{G}_1 \vee \mathcal{G}_2$ is the smallest sigma algebra containing both $\mathcal{G}_1$ and $\mathcal{G}_2$.

Solution: From the definition of conditional expectation, for some $\mathcal{G}$-measurable r.v. Y we have

$$\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A] \;\; \forall A \in \mathcal{G}$$

then $Y = E[X|\mathcal{G}]$.

Let $Y = \mathbb{E}[X|\mathcal{G}_2]$. To prove the claim we need to show

$$\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A] \;\; \forall A \in \mathcal{G}_2 \vee \mathcal{G}_3.$$

Define $\Lambda = \{A \in \mathcal{G}_2 \vee \mathcal{G}_3 : \mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A]\}$. From this definition it follows that Λ is a λ-system (this follows from linearity of expectation). Now consider the set $\Pi = \{G \cap H : G \in \mathcal{G}_2, H \in \mathcal{G}_3\}$. Clearly $\Pi \subset \Lambda$ and $\sigma(\Pi) = \mathcal{G}_2 \vee \mathcal{G}_3$. Then by the $\pi - \lambda$ theorem, to prove the claim it is sufficient to show that $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A] \;\; \forall A \in \Pi$.

Clearly for any $G \in \mathcal{G}_2, H \in \mathcal{G}_3$ we have $\mathbf{1}_H$ is independent of $X\mathbf{1}_G$ and $Y\mathbf{1}_G$ therefore

$$\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_{G\cap H}] = \mathbb{E}[X\mathbf{1}_G\mathbf{1}_H] = \mathbb{E}[X\mathbf{1}_G]\mathbb{E}[\mathbf{1}_H]$$

and

$$\mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_{G\cap H}] = \mathbb{E}[Y\mathbf{1}_G\mathbf{1}_H] = \mathbb{E}[Y\mathbf{1}_G]\mathbb{E}[\mathbf{1}_H]$$

Also, $\mathbb{E}[X\mathbf{1}_G] = \mathbb{E}[Y\mathbf{1}_G]$, because $Y = \mathbb{E}[X|\mathcal{G}_2]$ and $G \in \mathcal{G}_2$. therefore

$$\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A]$$

Since this holds for an arbitrary $A \in \Pi$, it holds for all Π, and we are done. We conclude

$$\mathbb{E}[X|\mathcal{G}_2 \vee \mathcal{G}_3] = Y = \mathbb{E}[X|\mathcal{G}_2].$$

2. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_n)_{n\geq 0}$ is a filtration, and $(A_n)_{n\geq 0}$ is a non-decreasing sequence of random variables such that

   a) $A_0 = 0$
   b) A_n is $\mathcal{F}_n$ measurable
   c) $\mathbb{E}[A_n^2]$ is finite.

   Also assume that $(B_n)_{n\geq 0}$ is a sequence random variables such that

   i) $0 < \mathbb{E}[B_n^2] < \infty$ and $\mathbb{E}[B_n] = 0$ for any $n \geq 0$

ii) B_n is $\mathcal{F}_n$ measurable

iii) B_n is independent of $\mathcal{F}_{n-1}$ for each $n \geq 1$.

(a) Show that if $(M_n)_{n\geq 0}$ is a square integrable martingale such that $M_0 = 0$ and $(M_n^2 + A_n)_{n\geq 0}$ is a supermartingale, then $M_n = A_n = 0$ almost surely for any $n \geq 0$.

(b) If A_n is $\mathcal{F}_{n-1}$ measurable for each $n \geq 1$, find a martingale $(M_n)_{n\geq 0}$ such that $(M_n^2 - A_n)_{n\geq 0}$ is a martingale.

Solution:

(a) Since $(M_n)_{n\geq 0}$ is a martingale, we know have

$$\begin{aligned}\mathbb{E}[M_{n+1}^2|\mathcal{F}_n] &= \mathbb{E}[(M_{n+1} - M_n)^2|\mathcal{F}_n] + 2M_n\mathbb{E}[(M_{n+1} - M_n)|\mathcal{F}_n] + \mathbb{E}[M_n^2|\mathcal{F}_n] \\ &= M_n^2 + \mathbb{E}[(M_{n+1} - M_n)^2|\mathcal{F}_n]\end{aligned}$$

Since, $M_n^2 + a_n$ is a supermartingale, we get that

$$\mathbb{E}[M_{n+1}^2 + A_{n+1}|\mathcal{F}_n] \leq M_n^2 + A_n$$

and thus combining this with the above we obtain that

$$\mathbb{E}[A_{n+1} - A_n|\mathcal{F}_n] + \mathbb{E}[(M_{n+1} - M_n)^2|\mathcal{F}_n] \leq 0.$$

Integrating this yields that

$$\mathbb{E}[A_{n+1} - A_n] + \mathbb{E}[(M_{n+1} - M_n)^2] \leq 0.$$

Since $A_{n+1} \geq A_n$ we conclude that almost surely, $M_{n+1} = M_n$ and $A_{n+1} = A_n$. Induction finishes the proof.

(b) For the second part, we can construct the martingale in the following form:

$$M_n = \sum_{k=1}^{n} B_k C_k$$

where C_k we choose to be $\mathcal{F}_{k-1}$ measurable. The martingale condition is then automatically satisfied because B_k is independent of $\mathcal{F}_{k-1}$ and has mean 0. In order to satisfy the second property, notice that

$$\begin{aligned}\mathbb{E}[M_n^2 - A_n|\mathcal{F}_{n-1}] &= M_{n-1}^2 - A_n + \mathbb{E}[(M_n - M_{n-1})^2|\mathcal{F}_{n-1}] = M_{n-1}^2 - A_n + \mathbb{E}[B_n^2 C_n^2|\mathcal{F}_{n-1}] \\ &= M_{n-1}^2 - A_n + C_n^2\mathbb{E}[B_n^2].\end{aligned}$$

Thus, if we want this to be equal to $M_{n-1}^2 - A_{n-1}$, then we need to choose C_n such that

$$C_n^2\mathbb{E}[B_n^2] = A_n - A_{n-1}$$

which is possible with

$$C_n = \sqrt{\frac{A_n - A_{n-1}}{\mathbb{E}[B_n^2]}}.$$

3. Assume that $X(t)$ is a simple Poisson process. Find the joint distribution of $(X(t_1), X(t_2))$ and then the conditional expectation $\mathbb{E}[X(t_1)|X(t_2)]$.

Solution: We have to distinguish two cases, one is $t_1 \geq t_2$ and $t_2 > t_1$.

In the case $t_1 \geq t_2$, it si easy to see from independence of increments that

$$\mathbb{E}[X(t_1)|X_{t_2}] = \mathbb{E}[X_{t_1} - X_{t_2}|X_{t_2}] + X_{t_2} = X_{t_2} + (t_1 - t_2).$$

For the case of $t_1 < t_2$ we need the joint distribution, which we can find using the fact that

$$\mathbb{P}(X(t_2) = k + n, X(t_1) = n) = \frac{e^{-(t_2-t_1)}((t_2 - t_1))^k}{k!} \frac{e^{-t_1}(t_1)^n}{n!} \quad k, n \in \mathcal{N}$$

Therefore the conditional distribution of $(X(t_1)|X(t_2))$ is

$$\mathbb{P}(X(t_1) = n|X(t_2) = k + n) = \frac{\frac{e^{-(t_2-t_1)}((t_2-t_1))^k}{k!} \frac{e^{-t_1}(t_1)^n}{n!}}{\frac{e^{-t_2}(t_2)^{(k+n)}}{(k+n)!}} \quad k, n \in \mathcal{N}$$

$$= \binom{k+n}{n} \left(\frac{t_1}{t_2}\right)^n \left(1 - \frac{t_1}{t_2}\right)^k \quad k, n \in \mathcal{N}$$

So the distribution of $X(t_1)$ conditioned on $X(t_2)$ is a Binomial distribution with parameters $(X(t_2), \frac{t_1}{t_2})$, which then implies

$$\mathbb{E}[X(t_1)|X(t_2)] = \frac{t_1}{t_2} X(t_2).$$

4. (a) Let X be a real-valued r.v. on a probability space $\Omega, \mathcal{F}, \mathbb{P}$ with density $f(x) = \frac{1}{3}\mathbb{I}_{[0,3]}(x)$. Find the correct assertions.

   i. $\mathbb{P}(X \in (0, 3)) = 1$.
   ii. For all $\omega \in \Omega$, $X(\omega) \in (0, 3)$.
   iii. For all $\omega \in \Omega$, $X(\omega) \in [0, 3]$.

   (b) Let $(X_n)_n$ be a sequence of real-valued random variables. Find and justify the correct assertions.

   i. $\{\sup_{n\geq1} X_n < \infty\}$ is an asymptotic event.
   ii. $\{\sup_{n\geq1} X_n < c\}$ for some $c \in \mathbb{R}$ is an asymptotic event.

Solution: (a) $\mathbb{P}(X \in (0, 3)) = 1$; (b) $\{\sup_{n\geq1} X_n < \infty\}$ is an asymptotic event.

5. Let X be a r.v. with Cauchy distribution $\mathcal{C}(1)$ (that means with density $f(x) = \frac{1}{\pi(1+x^2)}$ w.r.t. the Lebesgue measure on $\mathbb{R}$).

a) Determine the density of $Z = X^{-1}$.

b) Determine the density f of $\log |X|$.

Solution: (a) Z also follows Cauchy distribution $\mathcal{C}(1)$. (b) $Z = log|X|$ admits p.d.f. $f(z) = \frac{1}{\pi cosh(z)}$.

6. Let $(X_n)_n$ be a sequence of real-valued random variables with respective densities $f_n(x) = \frac{n^2}{2} e^{-n^2|x|}$.

   (a) Compute for all $n \in \mathbb{N}^*$ $\mathbb{P}(|X_n| > n^{-3/2})$

   (b) Compute $\mathbb{P}(\limsup\{|X_n| > n^{-3/2}\})$

   (c) What is the probability that $\sum_n X_n$ converges absolutely?

Solution: (a) We have $\mathbb{P}\left(\{|X_n| > n^{-3/2}\}\right) = \int_{|x|>n^{-3/2}} f_n(x)dx = e^{-\sqrt{n}}$. (b) Since

$$\sum_{n\geq 1} e^{-\sqrt{n}} < +\infty,$$

Borel-Cantelli's Lemma gives $\mathbb{P}\left(\text{limsup}\{|X_n| > n^{-3/2}\}\right) = 0$. (c) Set $A = \text{limsup}\{|X_n| > n^{-3/2}\}$. Thus, we have $A^c = \text{liminf}\{|X_n| \leq n^{-3/2}\}$. For any $w \in A^c$, there exists an integer n_w such that, for all $n \geq n_w$, $w \in \{|X_n| \leq n^{-3/2}\}$. Therefore the series $\sum_{n\geq 1} X_n(w)$ converges absolutely for any $w \in A^c$ and A^c is of probability 1.

7. Let $(X_n)_n$ be a sequence of independent and identically distributed real-valued random variables. Show that $\frac{X_n}{n}$ converges almost surely to 0 if and only if X_1 is integrable.

Solution: Recall that X_n/n converges a.s. to 0 if and only if for any $\epsilon > 0$, we have $\mathbb{P}\left(\text{limsup}\{|X_n|/n > \epsilon\}\right) = 0$. Since the X_n are independent, this is equivalent to

$$\sum_{n\geq 1} \mathbb{P}\left(|X_n|/n > \epsilon\right) < \infty$$

for any $\epsilon > 0$. Next, since the X_n are identically distributed, we have $\mathbb{P}\left(|X_n|/n > \epsilon\right) = \mathbb{P}\left(|X_1|/n > \epsilon\right)$ for any $n \geq 1$ and $\epsilon > 0$. The previous condition is equivalent to $\sum_{n\geq 1} \mathbb{P}\left(|X_1|/n > \epsilon\right) < \infty$ for any $\epsilon > 0$, which is equivalent in turn to $\mathbb{E}[|X_1|/\epsilon] < \infty$ for any $\epsilon > 0$ since $\mathbb{E}[|X_1|/\epsilon] = \int_0^\infty \mathbb{P}\left(|X_1|/\epsilon > t\right) dt$.

8. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables with standard Gaussian distribution $N(0,1)$. We recall that $\mathbb{E}[e^{X_1}] = e^{\frac{1}{2}}$. For all $n \geq 1$, set $S_n = \sum_{i=1}^n X_i$ and $M_n = e^{S_n - \frac{n}{2}}$.

(a) Justify the a.s. convergence of $\frac{S_n}{n}$ and determine the limit.

(b) Show that $M_n \to 0$ a.s. as $n \to +\infty$.

(c) For any $n \geq 1$, compute $\mathbb{E}[M_n]$.

(d) Do we have $M_n \to 0$ in L_1? Justify your answer.

(e) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $\sum_n a_n^2 < \infty$. Show that $\sum_{n\geq 1} a_n X_n$ converges to a random variable a.s. and in L^2.

Solution: (a) $(X_n)_n$ is an i.i.d. sequence of integrable random variables. Thus the strong law of large numbers gives $\frac{S_n}{n} \to \mathbb{E}[X_1] = 0$ a.s. (b) This is a straightforward consequence of (a) since $M_n = e^{n\left(\frac{S_n}{n} - 1/2\right)}$. (c) Since the sequence $(X_n)_n$ is i.i.d., we have $\mathbb{E}[M_n] = \left(\mathbb{E}[e^{X_1}]\right)^n e^{-\frac{n}{2}} = 1$. (d) We proceed by contradiction. Assume that M_n converges in L_1 to a random variable M. On the one hand, this implies that $\mathbb{E}[M_n] \to \mathbb{E}[M]$ as $n \to \infty$. In view of (c), we then have $\mathbb{E}[M] = 1$. On the other hand, there exists a subsequence of M_n that converges almost surely to M. Since M_n converges almost surely to 0, this implies that $M = 0$ a.s. This contradicts that $\mathbb{E}[M] = 1$. Therefore M_n does not converge in L_1 (e) We have for any integer $N \geq 1$ that $\sum_{n=1}^N a_n X_n \sim N(0, \sum_{k=1}^N a_n^2)$. By assumption the series $\sum_n a_n^2$ converges to $\sigma^2 = \sum_{n\geq 1} a_n^2$. Thus, $\sum_{n=1}^N a_n X_n$ converges in distribution to a random variable $Z \sim N(0, \sigma^2)$. Levy's theorem guarantees that $\sum_{n=1}^N a_n X_n$ also converges to Z a.s. and in L^2.