Topology Comprehensive Exam Questions

1. Let $S$ be a knot in $\mathbb{R}^3$, i.e. an embedded submanifold diffeomorphic to circle. Let $C = \mathbb{R}^3 \setminus S$, the complement of $S$ in $\mathbb{R}^3$. Show that there is a 1-form on $C$ that is not exact.

**Solution:** If $B$ be a small ball centered at a point of the knot, then there is a diffeomorphism $\phi\colon B \to \mathbb{R}^3$ taking $S \cap B$ to the $z$-axis. Consider the standard angle form $d\theta$ on the complement of the $z$-axis given by

$$d\theta = \frac{xdy - ydx}{x^2 + y^2},$$

Consider a smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ that is identically 1 on the unit ball of radius 2 about the origin, and that vanishes outside the ball of radius 3. Then $fd\theta$ is not exact on the complement of the $z$-axis because its integral over the unit circle in the $xy$-plane is 1, while exact forms integrate to 0 along closed smooth loops. The form $\phi^*(fd\theta)$ extends to a 1-form on $C$ by setting it equal to 0 outside $B$. It is still not exact, because it restricts to a non-exact form on $B$.

**Remark** It is possible but considerably harder to arrange the form to be closed but not exact.

2. Let $M$ be a smooth manifold, and $f\colon M \to \mathbb{R}$ is a continuous positive finction. Find a smooth positive function $f_0\colon M \to \mathbb{R}$ such that $f_0 < f$.

**Solution:** We know there is a locally finite open cover $\{W_i\}$ by precompact open sets and let $\phi_i$ be the partition of unity subordinate to this cover. Let $m_i$ be the minumum of $f$ on $\bar{W}_i$; note that $m_i > 0$ because $f > 0$ and $\bar{W}_i$ is compact. Thus $\phi_i f \geq \phi_i m_i$ for each $i$. Set $f_0 := \frac{1}{2}\sum_i \phi_i m_i$. Then $f_0 < f = \sum_i \phi_i f$, and $f_0$ is a smooth positive function because any $x \in M$ has a neighborhood $U$ that lies intersects only finitely many $W_i$'s, say $W_1, \dots W_k$, so $f_0|_U = \sum_{j=1}^k \phi_j m_j$, so locally $f_0$ is a sum of positive smooth functions.

**Remark.** Replacing minimum by maximum we get smooth $f_1 > f$. Choosing the cover fine enough, we can actually show that $f_0, f_1$ are close to $f$.

3. Suppose $q\colon M \to N$ is a submersion, and $X$ is a vector fields on $N$. Show that there is a vector field $\hat{X}$ on $M$ such that $\hat{X}$ and $X$ are $q$-related (that is $dq(\hat{X}(p)) = X(q(p))$ for all $p \in M$).

**Solution:** Submersions are open maps, so for any open $U$ in $M$ we have that $q(U)$ is an open subset of $N$. By definition of a submersion and by Rank Theorem there is an open cover $\{U_\alpha\}$ of $M$ and there are diffeomorphisms $\psi_\alpha\colon \mathbb{R}^m \to U_\alpha$ and $\phi_\alpha\colon q(U_\alpha) \to \mathbb{R}^n$ such that $\phi_\alpha \circ q \circ \psi_\alpha\colon \mathbb{R}^m \to \mathbb{R}^n$ is the standard projection forgetting the last $k$ coordinates, where $k = m-n$. Use $\psi_{\alpha *}$ and $(\phi_\alpha)_*^{-1}$ to push the standard coordinate vector fields from the Euclidean space to $U_\alpha$ and $q(U_\alpha)$, and denote the resulting coordinate vector fields by $\frac{\hat\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i}$, respectively. Note that $\pi_*$ maps first $n$ coordinate vector fields to themselves, and the other $k$ coordinate vector fields to zero. Hence the same holds for $q$, i.e. $q_*(\frac{\hat\partial}{\partial y_i}) = \frac{\partial}{\partial y_i}$ for $i \le n$ and $q_*(\frac{\hat\partial}{\partial y_i}) = 0$ for $i > n$. Write $X|_{q(U_\alpha)} = \sum_{i=1}^n x_i^\alpha \frac{\partial}{\partial y_i}$, for (uniquely determined and necessarily smooth) functions $x_i^\alpha\colon q(U_\alpha) \to \mathbb{R}$. Define a vector field $\hat X_\alpha$ on $U_\alpha$ by $\hat X_\alpha = \sum_{i=1}^n (x_i^\alpha \circ q)\frac{\hat\partial}{\partial y_i}$. Then $q_*\hat X_\alpha = X|_{q(U_\alpha)}$.

Let $\{U_j\}$ be a locally finite countable subcover of $\{U_\alpha\}$ and let $f_j$ be the corresponding partition of unity. So $f_j \hat X_j$ is a (smooth) vector field on $M$. Define $\hat X = \sum_j f_j \hat X_j$; this is a smooth vector field on $M$, and for any $p \in M$ we check that $\hat X$ and $X$ are $q$-related:

$$(q_{*p}(\hat X(p)) = q_{*p}(\sum_j f_j(p)\hat X_j(p)) = \sum_j f_j(p) q_{*p}(\hat X_j(p)) =$$

$$= \sum_j f_j(p) X(q(p)) = 1 \cdot X(q(p)) = X(q(p)).$$

4. Suppose $f\colon M \to N$ is a map, and $S$ is an embedded submanifold of $N$ such that for each $x \in f^{-1}(S)$ the subspaces $T_{f(x)}S$ and $f_*(T_xM)$ span $T_{f(x)}N$. (In this case we say that $f$ is *transverse to* $S$). Denote dimensions of $M$, $N$, $S$ by $m, n, s$, respectively. Show that $f^{-1}(S)$ is an embedded submanifold of $M$ of dimension $m + s - n$.

**Solution:** Fix $x_0 \in f^{-1}(S)$. The notion of a submanifold is local, so we need to find an open neighborhood $U$ of $x_0$ such that $f^{-1}(S) \cap U$ is an embedded submanifold of $U$. Since $S$ is a submanifold, there is a neighborhood $V$ of $f(x_0)$ that is mapped by a diffeomorphism $\psi$ to $\mathbb{R}^n$ such that $\psi(V \cap S) = \mathbb{R}^s$. Let $\pi$ be the projection of $\mathbb{R}^n$ onto the orthogonal complement of $\mathbb{R}^s$, which will be denoted $\mathbb{R}^{n-s}$. Since $T_{f(x)}S$ and $f_*(T_xM)$ span $T_{f(x)}N$ for any $x \in f^{-1}(S)$, and in particular, for any $x \in f^{-1}(S \cap V)$, their $\psi_*$-images span $\mathbb{R}^n$, and hence, the $\pi_*\psi_*$ image of $f_*(T_xM)$ spans the tangent space of $\mathbb{R}^{n-s}$ at 0. Hence 0 is a regular value for the map $\pi \circ \psi \circ f\colon f^{-1}(V) \to \mathbb{R}^{n-s}$. As $f^{-1}(V \cap S)$ is the preimage of 0 under the map, we conclude that $f^{-1}(V \cap S)$ is an embedded submanifold of $f^{-1}(V)$, which is a neighborhhod of $x_0$ in $M$. Finally, $f^{-1}(V \cap S) = f^{-1}(V) \cap f^{-1}(S)$, so $U := f^{-1}(V)$ is the desired neighborhood.

5. Let $M$ be the quotient of $S^2 \times S^1$ by the $\mathbb{Z}_2$-action given by $\iota(v, z) = (-v, \bar{z})$.
(a) Prove that the fundamental group of $M$ is the infinite diheadral group (the group of self-maps of $\mathbb{R}$ generated by two reflections, such as $a(t) = -t$ and $b(t) = 2 - t$).
(b) Prove that any continuous map from $M$ to $S^1$ is null-homotopic (you may use the Lifting Criterion as stated e.g. in Proposition 1.33 in Chapter 1 of Hatcher).

**Solution:** (a) Define two involutions on $S^2 \times \mathbb{R}$ by $A(v,t) = (-v, -t)$ and $B(v,t) = (-v, 2-t)$. Let $G$ be the group of homeomorphisms generated by $A$, $B$. Since $A^2 = 1 = B^2$, there are only four kinds of elements of $G$, namely $(AB)^k$, $(BA)^k$, $(AB)^k A$, $(BA)^k B$ where $k \in \mathbb{Z}$. Now $(AB)(v,t) = (v, t-2)$, and $BA(v,t) = (v, t+2)$, so $(AB)^k A(v,t) = (-v, -t-2k)$ and $(BA)^k B(v,t) = (-v, -t+2k+2)$. The induced $G$-action on the $\mathbb{R}$-coordinate is effective, i.e. no nontrivial element acts as identity on the $\mathbb{R}$-coordinate. Moreover, this $G$-action on $\mathbb{R}$ is that of an infinite diheadral group. Thus $G$ is isomorphic to the infinite diheadral group.

Given $(v_0, t_0)$ let $U$ be the product of an open hemisphere centered at $v_0$ with $(t_0 - 1, t_0 + 1)$. Then checking all for types of elements we see that $g(U)$ is disjoint from $U$ for all $g \in G$, so the $G$-action is wandering, so $S^2 \times \mathbb{R} \to (S^2 \times \mathbb{R})/G$ is a covering map.

Note that $S^2 \times \mathbb{R}$ is simply-connected because it is homotopy equivalent to $S^2$, which is simply-connected. Thus the fundamental group of $(S^2 \times \mathbb{R})/G$ is isomorphic to $G$.

It remains to show that $(S^2 \times \mathbb{R})/G$ is $M$. Let $G_0$ by the cyclic subgroup of $G$ generated by $AB$. Then $(S^2 \times \mathbb{R})/G_0$ is $S^2 \times \mathbb{R}/2\mathbb{Z}$, where the quotient maps $q\colon S^2 \times \mathbb{R} \to S^2 \times S^1$ takes $(v,t)$ to $(v, e^{\pi i t})$. Note that $q \circ A = \iota \circ q = q \circ B$ as

$$q(A(v,t)) = (-v, e^{-\pi i t}) = (-v, \overline{e^{\pi i t}}) = i(v, e^{\pi i t}) = (-v, \overline{e^{\pi i (2-t)}}) = q(B(v,t))$$

so $(S^2 \times \mathbb{R})/G$ is precisely the quotient of $S^2 \times S^1 = (S^2 \times \mathbb{R})/G_0$ by the $\mathbb{Z}_2$-action given by $\iota$, which is $M$.

(b) Since $\pi_1(M) = G$ is generated by elements of finite order and $\pi_1(S^1) = \mathbb{Z}$ has no elements of finite order, any homomorphism $\pi_1(M) \to \pi_1(S^1)$ is trivial, so by the lifting criterion any continuous map can be lifted to the cover $\mathbb{R} \to S^1$. Since $\mathbb{R}$ is contractible, any map $M \to \mathbb{R}$ is null-homotopic and composing it with $\mathbb{R} \to S^1$ we get a null-homotopy for the original map.

6. Show that homeomorphic topological manifolds have the same dimension.
(a) Show that any homeomorphism of a topological $n$-manifold onto a topological

$m$-manifold gives rise to a self-map of $S^{n-1}$ that is homotopic to identity and is a composition of maps $S^{n-1} \to S^{m-1}$ and $S^{m-1} \to S^{n-1}$ (Hint: consider small neighborhoods).
(b) Show that the existence of a map as in (a) implies $m = n$.

**Solution:** (a) Let $f$ be a homeomorphism of the $m$-manifold $M$ onto the $n$-manifold $N$. Fix $x \in M$, let $y := f(x)$, and consider a neighborhood $V$ of $y$ in $N$ such that there is a homeomorphism $\psi\colon V \to \mathbb{R}^n$ taking $y$ to 0. Since $f^{-1}(V)$ is a neigborhood of $x \in M$ there is a neighborhood $U \subset f^{-1}(V)$ of $x$ and a homeomorphism $\phi$ of $U$ onto $\mathbb{R}^m$ with $\phi(x) = 0$; we may also choose $U$ to have compact closure in $f^{-1}(V)$. Also $f(U)$ is open in $N$, so there is a neighborhood $W \subset f(U)$ of $y$, and we may assume that $\psi(W)$ is a round ball $B_\epsilon(0)$ around $0 \in \mathbb{R}^n$. Thus $\psi(f(U))$ is a neighorhood of 0 which contains $B_\epsilon(0)$, and $\psi(f(U))$ has compact closure. Consider concentric round spheres $S_R(0)$, $S_r(0)$ with $r < \epsilon$. The inclusion $\iota : S_r(0) \to \mathbb{R}^n \setminus \{0\}$ is homotopic to the map $v \to v\frac{R}{r}$ which is a homeomorphism between the two spheres (the most obvious is the straight line homotopy given by $F(t, v) := (1-t)v + tv\frac{R}{r}$ where $F\colon [0,1] \times S_r(0) \to \mathbb{R}^n \setminus \{0\}$; it does not vanish because no segment $[v, v\frac{R}{r}]$ passes through 0. On the other hand, $\iota$ factors through $\psi(f(U))\setminus\{0\}$ which is homeomorphic to $U\setminus\{x\}$, which in turn is homeomorphic to $\mathbb{R}^m\setminus\{0\} = S^{m-1}\times(0,1)$. Thus $\psi(f(U))\setminus\{0\}$ is homotopy equivalent to $S^{m-1}$, and pre/post composing the inclusions $S_r(0) \to \psi(f(U))\setminus\{0\}$, $\psi(f(U))\setminus\{0\} \to \mathbb{R}^n\setminus\{0\}$ with these homotopy equivalences we get continuous maps $S_r(0) = S^{n-1} \to S^{m-1}$ and $S^{m-1} \to S^{n-1} = S_R(0)$ whose composition is homotopic to a homeomorphism.

(b) Any continuous map $S^l \to S^k$ with $l < k$ is null-homotopic. This can be seen because such a map is homotopic to a smooth map that cannot be onto by Sard's theorem. Finally any non-sujective map between spheres are null-homotopic since once they miss a point one can assume (using stereographic coordinates) that the image of the map is in Euclidean space which is contractible. Thus if $n \neq m$, then one of the two maps above is null-homotopic, and hence so is their composition, but homeomorphisms are homotopy-equivalences so they are not null-homotopic.

7. Let $T$ be the torus $S^1 \times S^1$ and $f : S^1 \to T : \theta \mapsto= (\theta, p)$ for some point $p \in S^1$. Finally let $X$ be the space obtained by attaching a 2-cell $D^2$ to $T$ with the map $f$.

   (a) Compute the fundamental group of $X$.

   (b) Describe the universal cover of $X$. You may do this by drawing a picture but make sure the covering map is clear.

**Solution:** (a) To use Van Kampen's theorem let $A'$ be an open annular neighborhood of the image of $f$ in $T$ and $A$ be the union of $A'$ and the 2-cell $D^2$ in $X$. Also let $B'$ be the annular neighborhood of $\partial D^2$ in $D^2$ and $B$ be the union of $T$ and $B'$ in $X$. Notice that $X = A \cup B$ and $A \cap B$ retracts onto the circle $C = \text{image}(f)$. Similarly $A$ retracts to $D^2$ and $B$ retracts to $T$. Picking a base point $x_0$ on $C$ we know $\pi_1(A, x_0) \cong \pi_1(D^2, x_0) = 0$ and $\pi_1(A \cap B, x_0) = \pi_1(C, x_0) = \mathbb{Z}$. Let $i : (A \cap B) \to B$ be the inclusion map. We know $\pi_1(B, x_0) \cong \pi_1(T, x_0) \cong \mathbb{Z} \otimes \mathbb{Z}$ and the isomorphism can be choses so that $i_*(g)$ is a generator of the second factor of $\pi_1(B, x_0)$ where $g$ is a generator of $\pi_1(A \cap B, x_0) \cong \mathbb{Z}$. Now Van Kampen says

$$\pi_1(X, x_0) \cong \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\langle i_*(g)(j_*(g))^{-1} = e \rangle}$$

where $j : (A \cap B) \to A$ is the inclusion map. So clearly $i_*(g) = e$ in the free product. Thus we have

$$\pi_1(X, x_0) \cong \frac{(\mathbb{Z} \oplus \mathbb{Z}) * \{e\}}{\mathbb{Z}} = \mathbb{Z}.$$

(b) Let $R = S^1 \times \mathbb{R}$ and $f_i : S^1 \to R$ be given by $f_i(\theta) = (\theta, i)$ for $i \in \mathbb{Z}$. Now let $Y = R$ with a 2-cell $D_i^2$ glued to $R$ by $f_i$ for each $i$. We claim that $Y$ is the universal cover of $X$. To see this we first define the covering map $q : Y \to X$. We map $R \to T$ by $q(\theta, t) = (\theta, (\cos(2\pi t), \sin(2\pi t)))$ (here we are thinking of the second $S^1$ factor in $T$ as the unit circle in $\mathbb{R}^2$). Notice that $q \circ f_i = f$ if we choose $p = (1, 0)$. Thus thinking of the map $q$ as a map from $R$ to $X$ and defining $q$ on each $D_i^2$ to be the identify map $D_i^2 \to D^2$ we have a map form the disjoint union of $R$ and the $D_i^2$ to $X$ that descends to the quotient space $Y$. It is clear from construction the each point in $X$ is regularly covered in $Y$ so $Y$ is a covering space of $X$.

Moreover it is clear that $Y$ is simply connected by an argument similar to that given above. In particular attaching just one of the $D_i^2$ to $R$ will result in a space with trivial fundamental group. Then attaching further 2-cells will not add to the fundamental group. Thus $q : Y \to X$ is the universal cover of $X$.