Real Analysis Exam

[1] For $\varepsilon > 0$ and $k > 0$, denote by $A(k, \varepsilon)$ the set of $x \in \mathbb{R}$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{k\,|q|^{2+\varepsilon}} \qquad \text{for any integers } p, q \text{ with } q \neq 0.$$

Show that $\mathbb{R} \setminus \bigcup_{k=1}^{\infty} A(k, \varepsilon)$ is of Lebesgue measure zero.

[2] Fix an enumeration of all rational numbers: $r_1, r_2, r_3, \cdots$. For $x \in \mathbb{R}$, define

$$f(x) = \text{the cardinal number of the set } \{n \ : \ |x - r_n| \leq \frac{1}{2^n}\}.$$

(a) Show that f is Lebesgue measurable.

(b) Evaluate $\int_{\mathbb{R}} f(x)dx$.

[3] Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X (i.e., $\emptyset$, $X \in \mathcal{M}$ and $\mathcal{M}$ is closed under taking complements and countable unions of sets in $\mathcal{M}$).

(a) If μ is an extended real valued function on $\mathcal{M}$, what conditions must μ satisfy in order to be called a *measure*?

(b) Take $X = \mathbb{R}^n$ and let $\mathcal{M}$ be the set of *all subsets* of $\mathbb{R}^n$. Is $\mathcal{M}$ a σ-algebra?

(c) With X and $\mathcal{M}$ as in (b) above, let $d \in [0, n]$ and define d-dimensional Hausdorff measure $\mathcal{H}^d : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{H}^d(A) = \lim_{r \searrow 0} \left(\inf \left\{ \sum_{j=1}^{\infty} [\operatorname{diam}(A_j)]^d : A \subset \cup_{j=1}^{\infty} A_j, \operatorname{diam}(A_j) \le r \right\} \right). \tag{1}$$

Here $\operatorname{diam}(A_j) = \sup\{\|x - y\| : x, y \in A_j\}$ is the diameter of A_j. Show that the limit in (1), and hence $\mathcal{H}^d$, is well defined.

(d) Is $\mathcal{H}^1$ a measure? Justify your answer.

[4] Let $f : \mathbb{R} \to \mathbb{R}$ be in $L^1(\mathbb{R})$, and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function of period 1 and $\int_0^1 g(x)\,dx = 0$. Find

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)g(nx)\,dx.$$

Hint: You may use the fact that step functions are dense in $L^1(\mathbb{R})$.

[5] Let $f : [0,1] \to [0,1]$ be continuously differentiable and satisfy $f(0) = 0,\ f(1) = 1$.

(a) Show that the Lebesgue measure of

$$f\Big(\{x \in [0,1] : |f'(x)| < 1/m\}\Big)$$

is less than or equal to $1/m$.

(b) Use part (a) to show that there is at least one horizontal line $y = y_0 \in [0,1]$ which is nowhere tangent to the graph of f. Recall that the graph of f is $\{(x, f(x)) : x \in [0,1]\}$.

[6] Let X, Y, and Z be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be maps. Assume further that

- X is compact;
- f is surjective and continuous; and
- $g \circ f$ is continuous.

Show that g is continuous.

[7] Let H be a real Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$. Assume that $B : H \times H \to \mathbb{R}$ is bilinear (that is, $B(x, y)$ is linear in x for any fixed y and is linear in y for any fixed x). Assume further that there are positive constants C_1 and C_2 such that

$$\begin{aligned} |B(x,y)| &\leq C_1 \|x\| \, \|y\| \qquad x \in H, y \in H; \\ |B(x,x)| &\geq C_2 \|x\|^2 \qquad x \in H. \end{aligned}$$

(a) Show that there is a bounded linear operator $A : H \to H$ such that $B(x, y) = \langle Ax, y \rangle$ for all $x, y \in H$.

(b) Show that the operator A is one-to-one and onto.

[8] Let X be a complex Banach space, $I : X \to X$ denote the identity, and $S, T : X \to X$ be bounded linear operators. Denote by $\sigma(A) \subset \mathbb{C}$ the spectrum of operator A.

(a) Show that $I - ST$ has a bounded inverse if and only if $I - TS$ has a bounded inverse.

(b) Show that $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$.

(c) Show that $ST - TS \neq I$.

Algebra Exam

1. Let $n \geq 5$. Prove the following:

   (a) The only non-trivial normal subgroups of S_n is A_n.

   (b) S_n has no subgroup of index r, where $2 < r < n$.

   (c) List the normal subgroups of S_4.

2. (a) Let H be a proper subgroup of a finite group G. Show that G is not union of all conjugates of H.

   (b) Give an example of a group G, having a subgroup H, and an element a, such that $aHa^{-1} \subset H$, but $aHa^{-1} \neq H$.

3. A commutative ring A is called a Boolean ring if $x^2 = x$ for all $x \in A$.

   (a) Prove that if a Boolean ring contains no divisors of 0 it is either $\{0\}$ or is isomorphic to $\mathbb{Z}/(2)$. Deduce that in a Boolean ring every prime ideal is maximal.

   (b) Prove that in a Boolean ring every ideal $I \neq A$ is the intersection of the prime ideals containing I.

4. (a) Let R be a commutative ring with identity. Prove that every proper ideal I of R is contained in some maximal proper ideal.

   (b) Let k be a field, $R = k[x, y]$ and $I = (x^2 + y^2 - 1)$. Exhibit a maximal proper ideal containing I. Prove your claim.

5. Let f be a polynomial of degree n with coefficients in a field k of characteristic 0.

   (a) What is meant by a splitting field of f ?

   (b) Let L be a splitting field of f over k. Prove that $[L : k]$ is a divisor of $n!$.

6. Let F_q denote the finite field with q elements. For a prime p, consider the field F_{p^n} containing F_p as a subfield.

   (a) Prove that the group of automorphisms of F_{p^n} is cyclic of order n.

   (b) What is meant by a separable field extension ?

   (c) What is meant by a normal field extension ?

   (d) Is the field extension F_{p^n} over F_p separable and/or normal ?

7. Prove that a real quadratic form $Q(X_1, \ldots, X_n)$ can always be reduced to the form, $Q(X_1, \ldots, X_n) = \lambda_1 X_1^2 + \cdots + \lambda_n X_n^2$, with $\lambda_i \in \mathbb{R}$, using a linear change in co-ordinates.

8. Recall that $SL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) |\, \det(A) = 1\}$ and $sl(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) |\, \mathrm{tr}(A) = 0\}$. Prove that, $\exp(tA) \in SL(n, \mathbb{R})$ for all $t \in \mathbb{R}$ if and only if $A \in sl(n, \mathbb{R})$.

Real Analysis **Comprehensive Exam** **Fall 2002**

by XYC

Good luck!

[1] For $\varepsilon > 0$ and $k > 0$, denote by $A(k, \varepsilon)$ the set of $x \in \mathbb{R}$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{k\,|q|^{2+\varepsilon}} \qquad \text{for any integers } p, q \text{ with } q \neq 0.$$

Show that $\mathbb{R} \setminus \bigcup_{k=1}^{\infty} A(k, \varepsilon)$ is of Lebesgue measure zero.

Fix an arbitrary integer $L > 0$. We'll show that $[-L, L] \setminus \bigcup_{k=1}^{\infty} A(k, \varepsilon)$ is of measure zero. Let $k \geq 1$. For any $x \in [-L, L] \setminus A(k, \varepsilon)$, there are integers p, q $(q > 0)$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{kq^{2+\varepsilon}}.$$

We have

$$\left| \frac{p}{q} \right| \leq |x| + \left| x - \frac{p}{q} \right| \leq L + \frac{1}{kq^{2+\varepsilon}}.$$

Hence,

$$|p| \leq qL + \frac{1}{kq^{1+\varepsilon}} < qL + 1.$$

This shows

$$[-L, L] \setminus A(k, \varepsilon) \subset \bigcup_{q=1}^{\infty} \bigcup_{p=-qL}^{qL} \left(\frac{p}{q} - \frac{1}{kq^{2+\varepsilon}}, \frac{p}{q} + \frac{1}{kq^{2+\varepsilon}} \right),$$

and thus

$$\mu\Big([-L, L] \setminus A(k, \varepsilon)\Big) \leq \sum_{q=1}^{\infty} \sum_{p=-qL}^{qL} \frac{2}{kq^{2+\varepsilon}} = \frac{1}{k} \sum_{q=1}^{\infty} \frac{2(2qL+1)}{q^{2+\varepsilon}}.$$

The infinite series on the right hand side is convergent for $\varepsilon > 0$. It follows that

$$\mu\left([-L, L] \setminus \bigcup_{k=1}^{\infty} A(k, \varepsilon) \right) = \mu\left(\bigcap_{k=1}^{\infty} \Big([-L, L] \setminus A(k, \varepsilon) \Big) \right) \leq \inf_{k \geq 1} \left(\frac{1}{k} \sum_{q=1}^{\infty} \frac{2(2qL+1)}{q^{2+\varepsilon}} \right) = 0.$$

[2] Fix an enumeration of all rational numbers: $r_1, r_2, r_3, \cdots$. For $x \in \mathbb{R}$, define

$$f(x) = \text{the cardinal number of the set } \{r_n|\ |x - r_n| \leq \frac{1}{2^n}\}.$$

(a) Show that f is Lebesgue measurable.

(b) Evaluate $\int_{\mathbb{R}} f(x)dx$.

Part (a):

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the interval $[r_n - 2^{-n}, r_n + 2^{-n}]$:

$$f(x) = \begin{cases} 1 & |x - r_n| \leq 2^{-n}, \\ 0 & |x - r_n| > 2^{-n}. \end{cases}$$

Then, $\sum_{n=1}^{N} f_n$ are step functions and monotonically increases to the given function f as $N \to \infty$:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x).$$

Thus, the limit f is measurable.

Part (b):

Compute

$$\sum_{n=1}^{\infty} \|f_n\|_{L^1(\mathbb{R})} = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)dx = \sum_{n=1}^{\infty} \int_{r_n-2^{-n}}^{r_n+2^{-n}} 1\, dx = \sum_{n=1}^{\infty} 2^{1-n} = 2.$$

By Lebesgue's monotone convergence theorem (or by the completeness of $L^1(\mathbb{R})$), $f = \sum f_n$ is Lebesgue integrable and

$$\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)dx = 2.$$

Solutions

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3. Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X (i.e., $\phi, X \in \mathcal{M}$ and $\mathcal{M}$ is closed under taking complements and countable unions of sets in $\mathcal{M}$).

(a) If μ is an extended real valued function on $\mathcal{M}$, what conditions must μ satisfy in order to be called a *measure*?

Answer: One usually requires that μ be nonnegative, countably additive ($\mu(\cup A_j) = \sum \mu(A_j)$ where the A_j are disjoint sets), and satisfy $\mu(\phi) = 0$.

It is also acceptable to require only countable subadditivity ($\mu(\cup A_j) \le \sum \mu(A_j)$). This is sometimes called an *outer measure*.

(b) Take $X = \mathbb{R}^n$ and let $\mathcal{M}$ be the set of *all* subsets of $\mathbb{R}^n$. Is $\mathcal{M}$ a σ-algebra?

Answer: Yes clearly, since all conditions required of a σ-algebra involve nothing more than having certain sets in $\mathcal{M}$; all possible sets are in $\mathcal{M}$.

(c) With X and $\mathcal{M}$ as in (b) above, let $d \in [0, n]$ and define d-dimensional Hausdorff measure $\mathcal{H}^d : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{H}^d(A) = \lim_{r \searrow 0} \inf \left\{ \sum_{j=1}^{\infty} [\mathrm{diam}(A_j)]^d : A \subset \cup_{j=1}^{\infty} A_j,\ \mathrm{diam}(A_j) \le r \right\}. \qquad (1)$$

Here, $\mathrm{diam}(A_j) = \sup\{\|x - y\| : x, y \in A_j\}$ is the diameter of A_j. Show that the limit in (1), and hence $\mathcal{H}^d$, is well defined.
Solution: The infemum is a nondecreasing function of r. Therefore, the limit clearly exists. Technically, one could call the sets appearing after the lim inf something like $B(r)$ and observe that $B(r_1) \subset B(r_2)$ when $r_1 \le r_2$. The infemum of a subset of $B(r_2)$ must be at least as great as the infimum of $B(r_2)$.

(d) Is $\mathcal{H}^1$ a measure? Justify your answer.

Answer: According to the first definition, the answer is "no" for the following reason. One of the "big theorems" of real analysis, is that given any *translation invariant* measure on $\mathbb{R}$ for which the measure of an interval is its length, there exists a non-measurable set. Since we have defined $\mathcal{H}^d$ on all subsets, and it's easy to check that $\mathcal{H}^d$ is translation invariant, we do not have a measure, as long as the measure of an interval is its length (actually any finite nonzero number). It is easily checked that this holds for $\mathcal{H}^1$.

On the other hand, if you take the second definition (outer measure), then $\mathcal{H}^d$ is one, and one has more work to do. First of all, $\mathcal{H}^d_r = \inf B(r)$ is a measure. The only thing to check, really, is subadditivity on an arbitrary sequence of sets A_j. Let $\{C_{jk}\}_k$ be any countable cover of A_j by sets with diameter less than r. Since the doubly indexed collection $\{C_{jk}\}_{k,j}$ covers the union, we have

$$\mathcal{H}^d_r(\cup A_j) \leq \sum_k \sum_j \left[\mathrm{diam}(C_{jk})\right]^d .$$

Notice that the left side doesn't depend on the C_{jk}. Thus, we can take infema over collections of $\{C_{jk}\}_k$ one j at a time to obtain

$$\mathcal{H}^d_r(\cup A_j) \leq \sum_j \mathcal{H}^d_r(A_j). \tag{2}$$

Since $\mathcal{H}^d_r$ satisfies (2), we can use the monotonicity of $\mathcal{H}^d_r = \inf B(r)$ in r to obtain

$$\mathcal{H}^d_r(\cup A_j) \leq \sum_j \mathcal{H}^d(\cup A_j).$$

Notice that the right side is independent of r. Taking the limit as $r \to 0$ gives the result.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be in $L^1(\mathbb{R})$, and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function of period 1 with $\int_0^1 g(x)dx = 0$. Find

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)g(nx)dx.$$

Hint: You may use the fact that step functions are dense in L^1.

Solution: This is a version of the Riemann-Lebesge Theorem.

Let $\epsilon > 0$. Let f_ϵ be a step function with

$$\int |f_\epsilon - f| < \epsilon,$$

and let $M > 0$ such that

$$\left| \int_{-M}^{M} f(x)\,dx - \int_{-\infty}^{\infty} f(x)\,dx \right| < \epsilon.$$

For every ϵ,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} f(x) g(nx)\,dx \right| &\leq \left| \int_{-M}^{M} f(x) g(nx)\,dx - \int_{-\infty}^{\infty} f_\epsilon g(x) g(nx)\,dx \right| \\ &\quad + \left| \int_{-M}^{M} f_\epsilon(x) g(nx)\,dx \right| \\ &\leq 2G\epsilon + \left| \int_{-M}^{M} f_\epsilon(x) g(nx)\,dx \right| \end{aligned}$$

where $G = \sup_{x\in\mathbb{R}} |g(x)|$.

We can write

$$f_\epsilon(x) = \sum_{i=1}^{k} a_i \chi_{[x_{i-1},x_i]}(x)$$

on $[-M, M]$, for some constants $a_1, \ldots, a_k$ where $x_0 = -M < x_1 < \cdots < x_k = M$. Then

$$\left| \int_{-M}^{M} f_\epsilon(x) g(nx)\,dx \right| \leq \sum_{i=1}^{k} |a_i| \left| \int_{x_{i-1}}^{x_i} g(nx)\,dx \right|.$$

Changing variables, we get

$$\begin{aligned} \left| \int_{x_{i-1}}^{x_i} g(nx)\,dx \right| &= \left| \frac{1}{n} \int_{nx_{i-1}}^{nx_i} g(\xi)\,d\xi \right| \\ &= \frac{1}{n} \left| \int_{nx_{i-1}}^{\lceil nx_{i-1} \rceil} g(\xi)\,d\xi + \int_{\lfloor nx_i \rfloor}^{nx_i} g(\xi)\,d\xi \right| \end{aligned}$$

where $\lceil\ \rceil$ and $\lfloor\ \rfloor$ are the "least integer greater than" and "greatest integer less than" functions respectively. Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \left| \int_{-M}^{M} f_\epsilon(x) g(nx)\,dx \right| &\leq k \max\{a_i\} \limsup_{n\to\infty} \left(\frac{1}{n} 2G \right) \\ &= 0. \end{aligned}$$

Thus, for every $\epsilon > 0$,

$$\limsup_{n\to\infty} \left| \int_{-\infty}^{\infty} f(x)g(nx)\,dx \right| \leq 2G\epsilon.$$

Since ϵ is arbitrary,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)g(nx)dx = 0.$$

5. Let $f : [0,1] \to [0,1]$ be continuously differentiable and satisfy $f(0) = 0$, $f(1) = 1$.
(a) Show that the Lebesgue measure of

$$f\Big(\{x \in [0,1] : |f'(x)| < 1/m\}\Big)$$

is less than or equal to $1/m$.
(b) Use part (a) to show that there is at least one horizontal line $y = y_0 \in [0,1]$ which is nowhere tangent to the graph of f. Recall that the graph of f is $\{(x, f(x)) : x \in [0,1]\}$.

Solution: (This is a special case of Sard's Theorem.)

We will show that $B = \{f(x) : x \in [0,1], f'(x) = 0\}$ has measure zero. (Note that any $y_0 \notin B$ satisfies the requirements of the problem since whenever $x \in [0,1]$ and $f(x) = y_0 \notin B$, we have $y_0 \in [0,1]$ and must have $f'(x) \neq 0$.)

We first show that $B = \cap_{m=1}^{\infty} B_m$ where $B_m = f(A_m)$ and $A_m = \{x \in [0,1] : |f'(x)| < 1/m\}$ is the set given in the hint. On the one hand, if $y \in B$, then $y = f(x)$ for some $x \in [0,1]$ with $f'(x) = 0$. Clearly, $x \in A_m$ for all m, so $B \subset \cap B_m$. On the other hand, if $y \in \cap B_m$, then $y = f(x_m)$ for some $x_m \in [0,1]$ with $f'(x_m) = 0$. Since $[0,1]$ is compact, we can take a converging subsequence $x_{m_j} \to x_0 \in [0,1]$ and by continuity $f(x_0) = y$ and $f'(x_0) = 0$. This means $y \in B$.

The estimate of the measure of $B_m = f(A_m)$ comes from the change of variables formula $\int_{f(A)} 1 = \int_A |f'|$. Strictly speaking, this only holds on sets where f' does not change sign, but we can split $f(A_m)$ into $\{f(x) : x \in [0,1], 0 \leq f'(x) < 1/m\}$ and $\{f(x) : x \in [0,1], -1/m \leq f'(x) \leq 0\}$, and we still get an inequality:

$$\mathcal{L}(B_m) = \mathcal{L}(f(A_m)) = \int_{f(A_m)} 1 \leq \int_{A_m} |f'| \leq 1/m.$$

Since $B_{m+1} \subset B_m$,

$$\mathcal{L}(B) = \lim_{m \to \infty} \mathcal{L}(B_m) = 0.$$

[6] Let X, Y, and Z be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be maps. Assume further that

- X is compact;
- f is surjective and continuous; and
- $g \circ f$ is continuous.

Show that g is continuous.

Proof 1: Supposing that g is discontinuous at $y \in Y$, we'll derive a contradiction. From the discontinuity, there is a point sequence

(*) $$y_n \to y \text{ in } Y,$$

but $g(y_n) \not\to g(y)$ in Z. By taking a subsequence if necessary, we may, without loss of generality, assume that

(**) $$d(g(y_n), g(y)) \geq \varepsilon_0 > 0 \qquad \text{for all } n,$$

where ε_0 is a positive constant.

Since f is surjective, for every y_n there is a point $x_n \in X$ such that $f(x_n) = y_n$. Since X is compact, we can subtract a convergent subsequence $\{x_{k_n}\}$: $x_{k_n} \to x$ in X. By the continuity of f and $g \circ f$, we have

(***) $$y_{k_n} = f(x_{k_n}) \to f(x),$$

(****) $$g(y_{k_n}) = g \circ f(x_{k_n}) \to g \circ f(x).$$

By (*) and (***), we get $y = f(x)$. Combined with (****), it follows that $g(y_{k_n}) \to g(y)$, contradicting the supposition (**).

Proof 2: Only need to show that for any closed subset $C \subset Z$, $g^{-1}(C)$ is closed in Y.

By the continuity of $g \circ f$, $(g \circ f)^{-1}(C)$ is a closed subset of X.

Since any closed subset of a compact space is compact, $(g \circ f)^{-1}(C)$ is compact.

Since the continuous image of a compact set is compact, $f\Big((g \circ f)^{-1}(C)\Big)$ is compact.

Since any compact subset of a Hausdorff space is closed, $f\Big((g \circ f)^{-1}(C)\Big)$ is closed in Y.

The surjectivity of f implies $f\Big((g \circ f)^{-1}(C)\Big) = g^{-1}(C)$.

Therefore, $g^{-1}(C)$ is a closed subset of Y.

[7] Let H be a real Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$. Assume that $B : H \times H \to \mathbb{R}$ is bilinear (that is, $B(x, y)$ is linear in x for any fixed y and is linear in y for any fixed x). Assume further that there are positive constants C_1 and C_2 such that

$$|B(x,y)| \leq C_1 \|x\| \, \|y\| \qquad x \in H, y \in H;$$
$$|B(x,x)| \geq C_2 \|x\|^2 \qquad x \in H.$$

(a) Show that there is a bounded linear operator $A : H \to H$ such that $B(x, y) = \langle Ax, y\rangle$ for all $x, y \in H$.

(b) Show that the operator A is one-to-one and onto.

Part (a): For any fixed $x \in H$, the correspondence $H \to \mathbb{R}$, $y \mapsto B(x, y)$ is a bounded linear functional with norm bound $\|B(x, \cdot)\| \leq C_1\|x\|$. By Riesz's representation theorem, there exists a unique $A(x) \in H$ such that

$$B(x, y) = \langle A(x), y\rangle \qquad \text{for all } y \in H. \tag{*}$$

This defines an operator $A : H \to H$.

Let's first show that A is linear. For any $x_1, x_2 \in H$, $c_1, c_2 \in \mathbb{R}$, and any $y \in H$, we have

$$\begin{aligned}
\langle A(c_1x_1 + c_2x_2), y\rangle &= B(c_1x_1 + c_2x_2, y) && \text{(by (*))} \\
&= c_1B(x_1, y) + c_2B(x_2, y) && \text{(since } B \text{ is bilinear)} \\
&= c_1\langle A(x_1), y\rangle + c_2\langle A(x_2), y\rangle && \text{(by (*))} \\
&= \langle c_1A(x_1) + c_2A(x_2), y\rangle && \text{(since the inner product is bilinear).}
\end{aligned}$$

Since $y \in H$ is arbitrary, it follows that $A(c_1x_1 + c_2x_2) = c_1A(x_1) + c_2A(x_2)$.

Next we prove the boundedness of A. For any $x \in H$, we have

$$\|Ax\|^2 = |\langle Ax, Ax\rangle| = |B(x, Ax)| \leq C_1\|x\| \, \|Ax\|,$$

or, equivalently, $\|Ax\| \leq C_1\|x\|$. Thus, A is a bounded operator and $\|A\| \leq C_1$.

Part (b): Injectivity: We shall show $\mathrm{Kernel}(A) = 0$. Let $Ax = 0$. We have

$$0 = |\langle Ax, x\rangle| = |B(x, x)| \geq C_2\|x\|^2.$$

Thus, $x = 0$.

Surjectivity: We need to show $\mathrm{Range}(A) = H$. Since A is continuous, $\mathrm{Range}(A)$ is a closed subspace of the Hilbert space H. It suffices to prove that the orthogonal complement of $\mathrm{Range}(A)$ is 0. Let x be in the orthogonal complement. Then

$$0 = |\langle Ax, x\rangle| = |B(x, x)| \geq C_2\|x\|^2.$$

Thus, $x = 0$.

[8] Let X be a complex Banach space, $I : X \to X$ denote the identity, and $S, T : X \to X$ be bounded linear operators. Denote by $\sigma(A) \subset \mathbb{C}$ the spectrum of operator A.

(a) Show that $I - ST$ has a bounded inverse if and only if $I - TS$ has a bounded inverse.

(b) Show that $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$.

(c) Show that $ST - TS \neq I$.

Part (a): By summetry, it suffices to consider the "if" part. Assuming that $I - TS$ has a bounded inverse, we shall prove that $I - ST$ has a bounded inverse too.

We show that the bounded operator $I + S(I - TS)^{-1}T$ gives the inverse of $I - ST$:

$$\begin{aligned}
&\big[I + S(I-TS)^{-1}T\big](I - ST)\\
&\quad = I - ST + S(I-TS)^{-1}T - S(I-TS)^{-1}TST\\
&\quad = I - ST + S(I-TS)^{-1}T + S(I-TS)^{-1}\big[-I + (I - TS)\big]T\\
&\quad = I - ST + S(I-TS)^{-1}T - S(I-TS)^{-1}T + S(I-TS)^{-1}(I-TS)T\\
&\quad = I, \qquad \text{(the 2nd term + the last term =0, and the 3rd term + 4th term =0)}
\end{aligned}$$

$$\begin{aligned}
&(I - ST)\big[I + S(I-TS)^{-1}T\big]\\
&\quad = I - ST + S(I-TS)^{-1}T - STS(I-TS)^{-1}T\\
&\quad = I - ST + S(I-TS)^{-1}T + S\big[-I + (I-TS)\big](I-TS)^{-1}T\\
&\quad = I - ST + S(I-TS)^{-1}T - S(I-TS)^{-1}T + ST\\
&\quad = I.
\end{aligned}$$

Part (b): For $c \in \mathbb{C} \setminus 0$, we have the following equivalence:

$$\begin{aligned}
c \in \sigma(TS) &\iff cI - TS = c\,(I - c^{-1}TS) \text{ has no bounded inverse}\\
&\iff I - c^{-1}TS \text{ has no bounded inverse}\\
&\iff I - S(c^{-1}T) = I - c^{-1}ST \text{ has no bounded inverse} \qquad \text{(by Part (a))}\\
&\iff cI - ST \text{ has no bounded inverse}\\
&\iff c \in \sigma(ST).
\end{aligned}$$

Part (c): Suppose that $ST - TS = I$. Since ST and TS are bounded operators in a complex Banach space X, $\sigma(ST)$ and $\sigma(TS)$ are nonempty compact sets.

If $0 \in \sigma(TS)$, then $1 \in \sigma(ST)$ since $ST = I + TS$. By part (b), we have $1 \in \sigma(TS)$. Using $ST = I + TS$ again, we see $2 \in \sigma(ST)$. Repeating this argument, we infer that all positive integers are in $\sigma(ST)$, contradicting the boundedness of ST.

If $0 \in \sigma(ST)$, a similar argument shows that all negative integers are in $\sigma(TS)$, a contradiction.

It remains to consider the case where $0 \notin \sigma(TS)$ and $0 \notin \sigma(ST)$. In this case, Part (b) implies $\sigma(TS) = \sigma(ST)$. Combined with the asssumption $ST = I + TS$, it follows that the nonempty set $\sigma(ST)$ has a translational invariance:

$$\sigma(ST) = 1 + \sigma(TS) = 1 + \sigma(ST).$$

In particular, $\sigma(ST)$ has to be unbounded. This contradicts the boundedness of ST.