Solutions for the Analysis Qualifying Exam, Fall 2003.

Solve 5 of the following seven problems.

(1) Let f be a continuous function on $[0,1]$ such that $f(0) = f(1)$. Show that for each positive integers N, there exists a number c so that $f(c + 1/N) = f(c)$.

SOLUTION Extend f periodically, and define $g(x) = f(x) - f(x - 1/N)$. If we can find x_1 and x_2 such that $g(x_1)$ is positive and $g(x_2)$ is negative, then by the Intermediate Value Theorem, $g(c) = 0$ for some c between $1/j$ and $1/k$. But this means that $f(c+1/N) = f(c)$.

This is good progress – all that remains is to find x_1 and x_2. We haven't yet used the fact that $f(0) = f(1)$, so lets use that.

Telescoping the sum,

$$\begin{aligned} \sum_{j=1}^{N-1} g(j/N) &= \sum_{j=1}^{N-1} \left(f(j/N) - f((j-1)/N)\right) \\ &= f(1) = f(0) = 0 \ . \end{aligned}$$

If each $g(1/j)$ is zero, we are done – take $c = 0$. Otherwise, since the sum is zero, there must be both positive and negative terms. Hence there exist j and k with $1 \le j < k \le N$ and $g(1/j)g(1/k) < 0$. This gives us our x_1 and x_2.

(2) (a) Let X be the Banach space $\mathcal{C}([0,1])$, and let B be the closed unit ball in X. Does every bounded linear function on X assume a maximum on B? If so, prove that this is the case. If not, give an example of a bounded linear function that does not assume its maximum on B.

(b) Let X be the Banach space $L^2([0,1])$, with respect to Lebesgue measure, and let B be the closed unit ball in X. Does every bounded linear function on X assume a maximum on B? If so, prove that this is the case. If not, give an example of a bounded linear function that does not assume its maximum on B.

SOLUTION For this problem, think of the Banach–Alaoglu Theorem. This says that if X^* is the dual to some Banach space X, then the unit ball in X^* is compact in the weak–$*$ topology.

In the case that X is $L^2([0,1])$, we can identify X and X^* in the usual way (Riesz representation), and so the unit ball is weakly compact. Since bounded linear functionals are continuous, and since continuous functions assume their maxima on compact sets, every bounded linear functional on $L^2([0,1])$ assumes its maximum.

Alternate proofs for the L^2 case could be based on the projection lemma.

Now for part **(a)**, $\mathcal{C}([0,1])$ is *not* the dual of a Banach space, and so we should look for a counter example. Linear functionals on $\mathcal{C}([0,1])$ are signed Borel measures, and we can look for a counter example of the form $\mathrm{d}\mu = f(x)\mathrm{d}x$ where $\mathrm{d}x$ denotes Lebesgue measure. That is, let

$$\Lambda_f(\phi) = \int_0^1 \phi(x) f(x) \mathrm{d}x \ .$$

We want to choose f so that $\|\Lambda_f\| = \int_0^1 |f(x)|\mathrm{d}x = 1$, but

$$\int_0^1 \phi(x)f(x)\mathrm{d}x < \max_{0\le x\le 1} |\phi(x)|$$

for all $\phi \in \mathcal{C}([0,1])$. Take

$$f(x) = \begin{cases} 1, & 0 \le x \le 1/2 \\ -1, & 1/2 < x \le 1 \ . \end{cases}$$

(3) SOLUTION A great many "wierd function" constructions are done with a "moving rescaled blip". This strategy, which is natural to try, works here.

Let μ be Lebesgue measure on R. A measurable function f on R is said to be *unbounded near every point* if for every n,

$$\mu(\{x \ : \ f(x) > n\} \cap U) > 0$$

for every open set U. Is there a function f that is unbounded near every point and such that $f \in L^p(R)$ for each p with $1 \le p < \infty$? If not, prove that there is no such function. If so, give a construction, and prove that the function you construct has the required properties.

Let $g(x)$ be any non negative continuous function such that $g(0) = 1$, $g(x) \le 1$ for all x, and $\int_R g(x)\mathrm{d}x = 1$. Let $\{q_n\}$ be some enumeration of the rationals, and for each positive integer n, define

$$g_n(x) = ng(e^n(x - q_n)) \ .$$

Then

$$\begin{aligned}\int_R |g_n(x)|p\mathrm{d}x &\le \int_R n^p|g(e^n x)|^p n\mathrm{d}x \\ &= n^p e^{-n}\int_R |g(y)|^p\mathrm{d}y \\ &\le n^p e^{-n}\int_R |g(y)|\mathrm{d}y \\ &\le n^p e^{-n}\end{aligned}$$

Therefore, $\|g_n\|_p \le ne^{-n/p}$.

Finally, let

$$f(x) = \sum_{n=1}^{\infty} g_n(x) \ .$$

Notice that f is an increasing limit of continuous functions, so that it is measurable. By Minkowski's inequality and the fact that $\sum_{n=1}^{\infty} ne^{-n/p} < \infty$, f belongs to L^p for each p. Finally, given any N there are only finite many rationals q at which $f(q) \le N$ – namely, the first N in our enumeration. Therefore, $f > N/2$ on an open interval about infinitely many rationals near each x and hence f is unbounded near every point.

(4) Let μ denote Lebesgue measure on the σ–algebra $\mathcal{M}$ of Lebesgue measurable subsets of $[0,1]$. Let $\mathcal{A}$ be a sub σ–algebra of $\mathcal{M}$. Show that for any $f \in L^1([0,1],\mathcal{M},\mu)$, there exists a unique $g \in L^1([0,1],\mathcal{A},\mu)$ such that

$$\int_A g \mathrm{d}\mu = \int_A f \mathrm{d}\mu$$

for all $A \in \mathcal{A}$.

SOLUTION This is a standard application of the Radon–Nikodymn Theorem. Define a measure ν on $([0,1],\mathcal{A})$ by

$$\nu(A) = \int_A f(x)\mathrm{d}x \ .$$

It is clearly absolutely continuous with respect to Lebesgue measure, and hence there is a function $g \in L^1([0,1],\mathcal{A},\mu)$ – the Radon–Nikodymn derivative – so that

$$\int_A g \mathrm{d}\mu = \int_A f \mathrm{d}\mu$$

for all $A \in \mathcal{A}$.

(5) (a) Show that the real valued function f on $[0,\infty)$ defined by $f(t) = t\ln t$ is convex.

(b) Prove that for all summable infinite sequences $\{a_i\}$ and $\{b_i\}$ of non negative numbers,

$$\sum_{i=1}^{\infty} a_i \ln\left(\frac{a_i}{b_i}\right) \geq \left(\sum_{i=1}^{\infty} a_i\right) \ln\left(\frac{\sum_{i=1}^{\infty} a_i}{\sum_{i=1}^{\infty} b_i}\right) \ .$$

(c) Let μ denote Lebesgue measure on the σ–algebra $\mathcal{M}$ of Lebesgue measurable subsets of $[0,1]$. Show that

$$f \mapsto \int_{[0,1]} f(x)\ln f(x)\mathrm{d}\mu$$

defines a function on $L^1([0,1],\mathcal{M},\mu)$ for f with values in $[0,\infty]$ that is convex and lower semicontinuous.

Note: In part **(c)** as posed, the "for f" preceding "with values in $[0,\infty]$" was omitted, which led to questions. However, f must be non negative for the logarithm to be defined, so hopefully it was clear that the reference was to f.

To show that the functional is lower semicontinuous, we must show that for every C, the subset B_C of L^1 consisting of non negative functions f such that

$$\int_{[0,1]} f(x)\ln f(x)\mathrm{d}\mu \leq C$$

is closed. Consider a sequence $\{f_n\}$ in B_C with $\lim_{n\to\infty} f_n = f$ in L^1. Pass to a subsequence that converges almost everywhere. Notice that for all n, x,

$$f(x)\ln f(x) + \frac{1}{e} \geq 0 \ ,$$

so By Fatou's Lemma,

$$\int_{[0,1]} f(x) \ln f(x) \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_{[0,1]} f_n(x) \ln f_n(x) \mathrm{d}\mu \leq C \ .$$

The convexity is clear from part **(a)**.
SOLUTION For **(a)**, we compute $f'(t) = 1 + \ln(t)$ and $f''(t) = 1/t$. So f is convex.

For **(b)** Let μ be the probability measure on the positive integers N defined by

$$\mu(j) = \frac{b_j}{\sum_{k=1}^{\infty} b_k} \ .$$

Let ϕ be the function on N defined by $\phi(j) = a_j/b_j$. Then by Jensen's inequality, and the convexity of f from part **(a)**,

$$\int_N f(\phi)\mathrm{d}\mu \geq f\left(\int_N \phi \mathrm{d}\mu\right) \ .$$

This reduces to the desired inequality.

(6) Let B be the subset of $L^2(R)$ such that

$$\int_R x^2|f(x)|^2\mathrm{d}x \leq 1 \qquad \text{and} \qquad \int_R k^2|\hat{f}(k)|^2\mathrm{d}k \leq 1 \ .$$

Here, $\hat{f}$ denotes the Fourier transform of f. Show that B is a bounded, closed convex set in $L^2(R)$.

SOLUTION Note first of all that

$$\int_{-\infty}^{-1} f^2\mathrm{d}x + \int_1^{\infty} f^2\mathrm{d}x \leq \int_{-\infty}^{\infty} x^2 f^2(x)\mathrm{d}x \ ,$$

so all we need to do is to show that

$$\int_{-1}^{1} f^2(x)\mathrm{d}x \leq C$$

for some C and all $f \in B$. The main idea here is that the second condition implies that f is differentiable, and therefore bounded on $[-1,1]$. Here are the details

Since $\int_R k^2|\hat{f}(k)|^2\mathrm{d}k \leq 1$, f is differentiable with a square integrable derivative, and

$$\int_R |f'(x)|^2\mathrm{d}x = \frac{1}{4\pi^2}$$

with one popular normalization of the Fourier transform. (The proof does not depend on which one you choose; the statement to be proved is qualitative, not quantitative).

Now let g be a smooth function such that $g(x) = 0$ for $x < -3$, $g(x) = 1$ for $-1 \leq x \leq 1$ and $g(x) = 0 for x \geq 3$. Suppose also that $0 \leq g(x), |g'(x)| \leq 1$ for all x. Then $(fg)' = f'g + gf'$ so that

$$\begin{aligned}
\int_{-\infty}^{\infty} |(fg)'|^2 \mathrm{d}x &\leq 2\int_{-\infty}^{\infty} g^2|f'|^2 \mathrm{d}x + 2\int_{-\infty}^{\infty} f^2|g'|^2 \mathrm{d}x \\
&\leq 2\int_{-\infty}^{\infty} |f'|^2 \mathrm{d}x + 2\int_{-\infty}^{-1} f^2 \mathrm{d}x + 2\int_{1}^{\infty} f^2 \mathrm{d}x \\
&\leq 2\int_{-\infty}^{\infty} |f'|^2 \mathrm{d}x + 2\int_{-\infty}^{\infty} x^2 f^2 \mathrm{d}x
\end{aligned}$$

But for any x with $-1 \leq x \leq 1$, we have from the Schwarz inequality that

$$|f(x)g(x)| = \left| \int_{-3}^{x} (fg)'(x)\mathrm{d}x \right| \leq \sqrt{x+3}\|(fg)'\|_2 \leq 2\|(fg)'\|_2$$

Combining the last results, we see that $f(x)$ is bounded on $[0,1]$ with a bound that holds uniformly in B. This of course implies a bound on $\int_{-1}^{1} f^2(x)\mathrm{d}x$.

To show that B is closed, let $\{f_n\}$ be a sequence in B with $\lim_{n\to\infty} f_n = f$ in L^2. Pass to a subsequence such that $\lim_{n\to\infty} f_n = f$ almost everywhere. Then by Fatou's Lemma,

$$\int_R x^2 f^2(x)\mathrm{d}x \leq \liminf_{n\to\infty} \int_R x^2 f_n^2(x)\mathrm{d}x \leq 1\ .$$

By the Plancheral Theorem, $\lim_{n\to\infty} \hat{f}_n = \hat{f}$ in L^2, so we can also pass to a subsequence such that $\lim_{n\to\infty} \hat{f}_n = \hat{f}$ almost everywhere. Again by Fatou's Lemma,

$$\int_R k^2 \hat{f}^2(k)\mathrm{d}k \leq \liminf_{n\to\infty} \int_R k^2 \hat{f}_n^2(k)\mathrm{d}k \leq 1\ .$$

To show that B is convex, just use the Schwarz inequality: Let f and g belong to B, and let $0 \leq t \leq 1$. Then

$$\begin{aligned}
&\int_R x^2((1-t)f(x) + tg(x))^2 = \\
&\int_R ((1-t)xf(x) + txg(x))^2 \leq \\
&(1-t)^2\|xf\|_2^2 + 2t(1-t)\langle xf, xg\rangle_{L^2} + t^2\|xg\|_2^2 \leq \\
&(1-t)^2\|xf\|_2^2 + 2t(1-t)\|xf\|_1\|xg\|_2 + t^2\|xg\|_2^2 \leq \\
&(1-t)^2 + 2(1-t)t + t^2 = 1
\end{aligned}$$

(7) Let f be a function on the closed unit square $0 \leq x, y \leq 1$ that is separately continuous. That is, for each x, $y \mapsto f(x,y)$ is continuous, and for each y, $x \mapsto f(x,y)$ is continuous. Is f necessarily bounded? If so, prove that this is the case. Otherwise, give a counterexample.

SOLUTION No, it is not. Separate continuity is much weaker than continuity in the two variables jointly. To give a counter example, use (what else?) a "moving scaled blip construction".

Let $g(t)$ be any continuous function on the line such that $g(t) = 0$ for $t \leq 1/3$, $g(1/2) = 1$ and $g(t) = 0$ for $t \geq 2/3$. Define $f(x, y)$ for all x and y in the unit square by

$$f(x,y) = \begin{cases} 0, & y = 0 \\ (1/y)g(x/y), & 0 < y1 \end{cases} .$$

Clearly, because g is continuous, for each fixed y, $x \mapsto f(x, y)$ is continuous. If we fix any x then $y \mapsto f(x, y)$ is clearly continuos away from $y = 0$. But

$$\lim_{y \to 0} f(x,y) = \lim_{y \to 0} (1/y)g(x/y) = 0$$

because $g(x/y) = 0$ for all y so small that $x/y > 2/3$. Hence we have continuity in y also at $y = 0$.

Now take $(x_n, y_n) = (1/n, 1/(2n))$. Then for all n,

$$f(x_n, y_n) = 2n \ .$$

Hence f is separately continuous and unbounded.