Comprehensive Exam, Fall 2004 (Analysis)

Problem 1: Prove or give a counterexample to the following statement: Every function $f : [0, +\infty) \to \mathbb{R}$ for which the improper Riemann integral

$$\int_0^\infty f(x)dx$$

is convergent is Lebesgue integrable on $[0, +\infty)$.

Solution:

The statement is not true. Consider a function

$$f(x) = \frac{\sin x}{x} \quad \text{for } x > 0$$

and $f(0) = 1$. It is easy to show that the improper Riemann integral of f is convergent. However the Lebesgue integrals

$$\int_{[0,\infty)} f^+, \quad \text{and} \int_{[0,\infty)} f^-$$

are equal to $+\infty$ so not only is f not integrable but the Lebesgue integral

$$\int_{[0,\infty)} f$$

is not even well defined.

Problem 2: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n : \Omega \to \mathbb{R}, n \geq 1$, and $g : \Omega \to \mathbb{R}$ be functions in $L^1(\mu)$ such that there exists a constant $C > 0$ such that

$$\int_\Omega |f_n| \, d\mu \leq C$$

for all $n \geq 1$. Suppose moreover that

$$\frac{1}{n} f_n^2 \leq g \quad \text{on } \Omega.$$

Show that

$$\int_\Omega \frac{1}{n} f_n^2 \, d\mu \to 0 \quad \text{as } n \to \infty.$$

Solution: Denote

$$A_n = \{x : |f_n(x)| \geq n^{\frac{1}{3}}\}.$$

Then

$$\int_\Omega \frac{1}{n} f_n^2 d\mu = \int_{A_n} \frac{1}{n} f_n^2 d\mu + \int_{\Omega \setminus A_n} \frac{1}{n} f_n^2 d\mu \leq \int_{A_n} g d\mu + \frac{1}{n} n^{\frac{2}{3}} \mu(\Omega).$$

But

$$\mu(A_n) \leq \frac{C}{n^{\frac{1}{3}}} \to 0 \quad \text{as } n \to \infty$$

and therefore, since $g \in L^1(\mu)$,

$$\int_{A_n} g d\mu \to 0 \quad \text{as } n \to \infty$$

which completes the proof. (The proof of the last convergence can be found in any standard textbook.)

Problem 3: Define

$$B = C([0,1]) = \{f : [0,1] \to \mathbb{R} : \text{ f is continuous}\}, \quad \|f\|_B = \max_{0 \leq x \leq 1} |f(x)|$$

$$C = C^\alpha([0,1]) = \{f : [0,1] \to \mathbb{R} : f \in B \text{ and } \|f\|_C = \|f\|_B + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < +\infty\},$$

for some $\alpha \in (0,1]$. It is well known that equipped with the norms $\|\cdot\|_B$, and $\|\cdot\|_C$, the spaces B, and C respectively are Banach spaces (normed vector spaces complete with respect to the norm metric). Determine if the unit ball is compact in the spaces B and C. Is the unit ball of C compact as a subset of B?

Solution: Recall that a metric space a set X is compact if and only if every sequence in X has a subsequence converging to an element of X.

The unit ball in B is not compact. For instance consider for $n \geq 1$ a sequence of continuous functions f_n such that $f_n(x) = 0$ if $x \notin (1/(n+1), 1/n)$, $f(1/2(1/(n+1) + 1/n)) = 1$, and $0 \leq f_n(x) \leq 1$ for $x \in (1/(n+1), 1/n)$. Then $\|f_n\|_A = 1$ but $\|f_n - f_m\|_A = 1$ if $n \neq m$. Therefore the sequence does not have a convergent subsequence.

The unit ball in C is also not compact. Consider the sequence of functions

$$f_n(x) = \begin{cases} \frac{x}{2} \quad \text{for } 0 \leq x \leq \frac{1}{2^{n+1}}, \\ \\ \frac{1}{2^{n+1}} - \frac{x}{2} \quad \text{for } \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}, \\ \\ 0 \quad \text{otherwise.} \end{cases} \tag{1}$$

It is easy to see that $\|f_n\|_C = \frac{1}{2^{n+2}} + \frac{1}{2}$ but if $n \neq m$ then $\|f_n - f_m\|_C \geq \frac{1}{2^{n+2}} + 1/2$ so the sequence does not have a convergent subsequence.

However the unit ball in C is compact in B. To see this let f_n be functions such that $\|f_n\|_C \leq 1$. Then the functions f_n are equibounded and equicontinuous and so by the Arzela-Ascoli Theorem there is a subsequence f_{n_k} that converges uniformly on $[0,1]$ to a continuous function f. Obviously

$$\|f\|_B = \lim_{k \to \infty} \|f_{n_k}\|_B.$$

It remains to show that $\|f\|_C \leq 1$. Let now $x \neq y$. Then

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} = \lim_{k\to\infty} \frac{|f_{n_k}(x) - f_{n_k}(y)|}{|x - y|^\alpha}$$

$$\leq \lim_{k\to\infty} \sup_{x\neq y} \frac{|f_{n_k}(x) - f_{n_k}(y)|}{|x - y|^\alpha} \leq \lim_{k\to\infty} (1 - \|f_{n_k}\|_B) = 1 - \|f\|_B.$$

Therefore $\|f\|_C \leq 1$.

Problem 4: Let $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that
(i) for each $t \in [a, b]$, the function $x \to f(t, x)$ is continuous,
(ii) for each $x \in \mathbb{R}^n$, the function $t \to f(t, x)$ is Lebesgue measurable.
Show that f is $\mathcal{L} \otimes \mathcal{B}$ measurable, where $\mathcal{L}$ is the class of Lebesgue measurable sets on $[a, b]$, $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$, and $\mathcal{L} \otimes \mathcal{B}$ is the product σ-algebra of $\mathcal{L}$ and $\mathcal{B}$.

Solution: Let $r_1, ..., r_n, ...$ be a dense subset of $\mathbb{R}^n$ (for instance a sequence of all points with rational coordinates). For each integer $m \geq 1$ define a function $f_n : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ by

$$f_m(t, x) = f(t, r_k) \quad \text{if } |x - r_k| < \frac{1}{m} \text{ but } |x - r_i| \geq \frac{1}{m} \text{ for } 1 \leq i < k.$$

Then, by (i), for every $(t, x) \in [a, b] \times \mathbb{R}^n$ $f_m(t, x) \to f(t, x)$ as $m \to \infty$. Since the limit of measurable functions is measurable it is enough to show that the functions f_m are $\mathcal{L} \otimes \mathcal{B}$ measurable. To this end choose an open set $U \subset \mathbb{R}^n$. Then

$$f_m^{-1}(U) = \bigcup_{m=1}^{\infty} \left\{(t, x) \in [a, b] \times \mathbb{R}^n : f(t, r_k) \in U, |x - r_k| < \frac{1}{m},\ |x - r_i| \geq \frac{1}{m} \text{ for } 1 \leq i < k.\right\}$$

$$= \bigcup_{m=1}^{\infty} \left(\{t \in [a, b] : f(t, r_k) \in U\} \times \left\{x \in \mathbb{R}^n : |x - r_k| < \frac{1}{m},\ |x - r_i| \geq \frac{1}{m} \text{ for } 1 \leq i < k.\right\}\right)$$

$$= \bigcup_{m=1}^{\infty} ((\text{set in } \mathcal{L}) \times (\text{set in } \mathcal{B})) \in \mathcal{L} \otimes \mathcal{B}.$$

Problem 5: Fix a prime number p. A rational number x can be represented by $x = p^\alpha \frac{k}{l}$ with k, l not divisible by p, and $\alpha \in \mathbb{Z}$ is defined uniquely. Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ by

$$|x|_p := p^{-\alpha}, \quad \text{and} \qquad |0|_p := 0.$$

(a) Show that $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ and conclude that $d(x, y) := |x - y|_p$ defines a metric on $\mathbb{Q}$.

(b) Show that in the completion of $\mathbb{Q}$ w.r.t. the above metric a series of rational numbers

$$\sum_{n\geq 0} a_n$$

converges if and only if $|a_n|_p \to 0$.

Solution: (a) write $x = p^{\alpha_1}k_1/l_1$ and $y = p^{\alpha_2}k_2/l_2$ (k_1, k_2, l_1, l_2 not divisible by p). We may assume $\alpha_1 \geq \alpha_2$. Then

$$|x+y|_p = |p^{\alpha_2}(p^{\alpha_1-\alpha_2}l_2k_1 + l_1k_2)/(l_1l_2)|_p = |p^{\alpha_2}|_p = p^{-\alpha_2}$$

since p does not divide the terms in parenthesis above. Note that $\max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$. Hence $d(x,z) = |x-y+y-z|_p \leq |x-y|_p + |y-z|_p$, proving the triangle inequality. Obviously d is symmetric and $d(x,y) = 0$ iff $x = y$.

(b) Consider the partial sums $S_N = \sum_{n=1}^{N} a_n$. Suppose S_N converges. Then S_N is a Cauchy sequence, hence for $\epsilon > 0$ there is N_ϵ such that

$$|a_N|_p = |S_N - S_{N-1}|_p \leq \epsilon$$

for $N \geq N_\epsilon$. So $|a_N|_p \to 0$ w.r.t. the metric d. Suppose now $a_n \to 0$, i.e. $|a_n|_p \to 0$. Then for $M, N \in \mathbb{N}$

$$|S_{N+M} - S_M|_p \leq \max\{|a_{N+1}|_p, \ldots.., |a_{N+M}|_p\} \to 0$$

Hence S_N is a Cauchy sequences which converges in the completion of $\mathbb{Q}$ w.r.t. the metric d.

Problem 6: Let (X,d) be a compact metric space and denote by $B_R(a) \subset X$ the closed ball of radius $R > 0$ centered at $a \in X$. Suppose μ is a positive Borel measure on X satisfying for some $\beta > 0$ and for all $r \in (0,1)$ and $a \in X$

$$c_1\, r^\beta \leq \mu(B_r(a)) \leq\; r^\beta,$$

with $c_1 > 0$ independent of r and a. Fix a point $a \in X$. Find all $\alpha \in \mathbb{R}$ for which $x \mapsto d(x,a)^\alpha$ is in $L^1(X, d\mu)$.

Solution: Only the case $\alpha < 0$ is interesting. Since $d(x,a)^\alpha$ is bounded and continuous away from a it suffices to check whether

$$\int_{B_1(a)} d(x,a)^\alpha\; d\mu(x) < \infty.$$

Let $\{R_k\}$ be a strictly monotone decreasing sequence of positive reals and denote by B_k the balls $B_{R_k}(a)$. We will choose R_k such that $B_k \setminus B_{k+1}$ has essentially the same measure as B_k. To achieve this we compute

$$\mu(B_k \setminus B_{k+1}) = \mu(B_k) - \mu(B_{k+1}) \geq c_1 R_k^\beta - R_{k+1}^\beta$$

Hence, if we choose $R_{k+1} = R_k\gamma, 0 < \gamma < 1$, the last term is $R_k^\beta(c_1 - \gamma^\beta)$ which by appropriate choice of γ equals $R_k^\beta c_1/2$. We set $R_k = \gamma^k, k = 0, 1, ...$ and write

$$\int_{B_1(a)} d(x,a)^\alpha \; d\mu(x) = \sum_{k\geq 0} \int_{B_k \setminus B_{k+1}} d(x,a)^\alpha \; d\mu(x)$$

On each "shell" $B_k \setminus B_{k+1}$ we may bound the integrand above by R_{k+1}^α and from below by R_k^α. Since the shells have μ-measure at most R_k^β we find that

$$\int_{B_1(a)} d(x,a)^\alpha \; d\mu(x) \leq \sum_k \gamma^{(k+1)\alpha} \; \gamma^{k\beta}.$$

The latter geometric series converges if $\alpha > -\beta$. Since we also have

$$\int_{B_k \setminus B_{k+1}} d(x,a)^\alpha \; d\mu(x) \geq \gamma^{k\alpha} \; \gamma^{k\beta} \; c_1/2.$$

the condition $\alpha > -\beta$ is also necessary.

Problem 7: Let H be a Hilbert space. Show that if $T : H \to H$ is symmetric, i.e. $\langle x, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in H$, then T is linear and continuous.

Solution: First we show that T is linear. Let $x_1, x_2 \in H$ then for all $y \in H$ we have $\langle T(x_1 + x_2), y\rangle = \langle x_1 + x_2, Ty\rangle = \langle x_1, Ty\rangle + \langle x_2, Ty\rangle = \langle Tx_1, y\rangle + \langle Tx_2, y\rangle = \langle Tx_1 + Tx_2, y\rangle$. Hence $T(x_1 + x_2) = Tx_1 + Tx_2$. Similarly one shows that T is homogeneous. To see that T is continuous we first note that by the closed graph theorem it suffices to show that the $graph(T)$ is closed. Let $(x_n, Tx_n) \in graph(T)$ be a sequence in $H \times H$ converging to $(x, y) \in H \times H$. We claim: $Tx = y$. To see this consider

$$\|Tx - y\|^2 = \langle y - Tx, y - Tx\rangle = \lim_n \langle Tx_n - Tx, y - Tx\rangle = \lim_n \langle x_n - x, T(y - Tx)\rangle = 0$$

Hence $Tx = y$.