Problem 1: Prove or give a counterexample to the following statement: Every function $f : [0, +\infty) \to \mathbb{R}$ for which the improper Riemann integral
\[
\int_{0}^{\infty} f(x)\,dx
\]
is convergent is Lebesgue integrable on $[0, +\infty)$.

Solution:
The statement is not true. Consider a function $f(x) = \frac{\sin x}{x}$ for $x > 0$ and $f(0) = 1$. It is easy to show that the improper Riemann integral of $f$ is convergent. However the Lebesgue integrals
\[
\int_{[0, \infty)} f^+, \quad \text{and} \quad \int_{[0, \infty)} f^-
\]
are equal to $+\infty$ so not only is $f$ not integrable but the Lebesgue integral
\[
\int_{[0, \infty)} f
\]
is not even well defined.

Problem 2: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n : \Omega \to \mathbb{R}, n \geq 1$, and $g : \Omega \to \mathbb{R}$ be functions in $L^1(\mu)$ such that there exists a constant $C > 0$ such that
\[
\int_{\Omega} |f_n| \,d\mu \leq C
\]
for all $n \geq 1$. Suppose moreover that
\[
\frac{1}{n} f_n^2 \leq g \quad \text{on} \, \Omega.
\]
Show that
\[
\int_{\Omega} \frac{1}{n} f_n^2 \,d\mu \to 0 \quad \text{as} \quad n \to \infty.
\]

Solution: Denote
\[
A_n = \{ x : |f_n(x)| \geq n^{\frac{3}{4}} \}.
\]
Then
\[ \int_{\Omega} \frac{1}{n} f_n^2 d\mu = \int_{A_n} \frac{1}{n} f_n^2 d\mu + \int_{\Omega \setminus A_n} \frac{1}{n} f_n^2 d\mu \leq \int_{A_n} g d\mu + \frac{1}{n^{\frac{2}{3}}} \mu(\Omega). \]

But
\[ \mu(A_n) \leq \frac{C}{n^{\frac{2}{3}}} \to 0 \quad \text{as} \quad n \to \infty \]

and therefore, since \( g \in L^1(\mu) \),
\[ \int_{A_n} g d\mu \to 0 \quad \text{as} \quad n \to \infty \]

which completes the proof. (The proof of the last convergence can be found in any standard textbook.)

Problem 3: Define
\[ B = C([0, 1]) = \{ f : [0, 1] \to \mathbb{R} : f \text{ is continuous} \}, \quad \| f \|_B = \max_{0 \leq x \leq 1} |f(x)| \]
\[ C = C^\alpha([0, 1]) = \{ f : [0, 1] \to \mathbb{R} : f \in B \text{ and } \| f \|_C = \| f \|_B + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < +\infty \}, \]
for some \( \alpha \in (0, 1] \). It is well known that equipped with the norms \( \| \cdot \|_B \) and \( \| \cdot \|_C \), the spaces \( B \) and \( C \) respectively are Banach spaces (normed vector spaces complete with respect to the norm metric). Determine if the unit ball is compact in the spaces \( B \) and \( C \). Is the unit ball of \( C \) compact as a subset of \( B \)?

Solution: Recall that a metric space a set \( X \) is compact if and only if every sequence in \( X \) has a subsequence converging to an element of \( X \).

The unit ball in \( B \) is not compact. For instance consider for \( n \geq 1 \) a sequence of continuous functions \( f_n \) such that \( f_n(x) = 0 \) if \( x \not\in (1/(n + 1), 1/n) \), \( f(1/2(1/(n + 1) + 1/n)) = 1 \), and \( 0 \leq f_n(x) \leq 1 \) for \( x \in (1/(n + 1), 1/n) \). Then \( \| f_n \|_A = 1 \) but \( \| f_n - f_m \|_A = 1 \) if \( n \neq m \). Therefore the sequence does not have a convergent subsequence.

The unit ball in \( C \) is also not compact. Consider the sequence of functions
\[ f_n(x) = \begin{cases} 
\frac{x}{2} & \text{for } 0 \leq x \leq \frac{1}{2n+1}, \\
\frac{1}{2n+1} - \frac{x}{2} & \text{for } \frac{1}{2n+1} \leq x \leq \frac{1}{2n}, \\
0 & \text{otherwise}. 
\end{cases} \tag{1} \]

It is easy to see that \( \| f_n \|_C = \frac{1}{2n+2} + \frac{1}{2} \) but if \( n \neq m \) then \( \| f_n - f_m \|_C \geq \frac{1}{2n+2} + 1/2 \) so the sequence does not have a convergent subsequence.

However the unit ball in \( C \) is compact in \( B \). To see this let \( f_n \) be functions such that \( \| f_n \|_C \leq 1 \). Then the functions \( f_n \) are equibounded and equicontinuous and so by the Arzela-Ascoli Theorem there is a subsequence \( f_{n_k} \) that converges uniformly on \([0, 1]\) to a continuous function \( f \). Obviously
\[ \| f \|_B = \lim_{k \to \infty} \| f_{n_k} \|_B. \]
It remains to show that $\|f\|_C \leq 1$. Let now $x \neq y$. Then

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} = \lim_{k \to \infty} \frac{|f_{n_k}(x) - f_{n_k}(y)|}{|x - y|^\alpha} \leq \lim_{k \to \infty} \sup_{x \neq y} \frac{|f_{n_k}(x) - f_{n_k}(y)|}{|x - y|^\alpha} \leq \lim_{k \to \infty} (1 - \|f_{n_k}\|_B) = 1 - \|f\|_B.$$ 

Therefore $\|f\|_C \leq 1$.

**Problem 4:** Let $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that

(i) for each $t \in [a, b]$, the function $x \to f(t, x)$ is continuous,

(ii) for each $x \in \mathbb{R}^n$, the function $t \to f(t, x)$ is Lebesgue measurable.

Show that $f$ is $\mathcal{L} \otimes \mathcal{B}$ measurable, where $\mathcal{L}$ is the class of Lebesgue measurable sets on $[a, b]$, $\mathcal{B}$ is the Borel $\sigma$-algebra on $\mathbb{R}^n$, and $\mathcal{L} \otimes \mathcal{B}$ is the product $\sigma$-algebra of $\mathcal{L}$ and $\mathcal{B}$.

**Solution:** Let $r_1, \ldots, r_n, \ldots$ be a dense subset of $\mathbb{R}^n$ (for instance a sequence of all points with rational coordinates). For each integer $m \geq 1$ define a function $f_m : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ by

$$f_m(t, x) = f(t, r_k) \quad \text{if} \quad |x - r_k| < \frac{1}{m} \quad \text{but} \quad |x - r_i| \geq \frac{1}{m} \quad \text{for} \quad 1 \leq i < k.$$ 

Then, by (i), for every $(t, x) \in [a, b] \times \mathbb{R}^n$ $f_m(t, x) \to f(t, x)$ as $m \to \infty$. Since the limit of measurable functions is measurable it is enough to show that the functions $f_m$ are $\mathcal{L} \otimes \mathcal{B}$ measurable. To this end choose an open set $U \subset \mathbb{R}^n$. Then

$$f_m^{-1}(U) = \bigcup_{m=1}^{\infty} \left\{ (t, x) \in [a, b] \times \mathbb{R}^n : f(t, r_k) \in U, |x - r_k| < \frac{1}{m}, |x - r_i| \geq \frac{1}{m} \text{ for } 1 \leq i < k \right\}$$

$$= \bigcup_{m=1}^{\infty} \left( \left\{ t \in [a, b] : f(t, r_k) \in U \right\} \times \left\{ x \in \mathbb{R}^n : |x - r_k| < \frac{1}{m}, |x - r_i| \geq \frac{1}{m} \text{ for } 1 \leq i < k \right\} \right)$$

$$= \bigcup_{m=1}^{\infty} \left( \text{(set in } \mathcal{L} \text{) } \times \text{(set in } \mathcal{B} \text{)} \right) \in \mathcal{L} \otimes \mathcal{B}.$$ 

**Problem 5:** Fix a prime number $p$. A rational number $x$ can be represented by $x = p^{\alpha_k/l}$ with $k, l$ not divisible by $p$, and $\alpha \in \mathbb{Z}$ is defined uniquely. Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ by

$$|x|_p := p^{-\alpha}, \quad \text{and} \quad |0|_p := 0.$$ 

(a) Show that $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ and conclude that $d(x, y) := |x - y|_p$ defines a metric on $\mathbb{Q}$. 
(b) Show that in the completion of $\mathbb{Q}$ w.r.t. the above metric a series of rational numbers

$$
\sum_{n \geq 0} a_n
$$

converges if and only if $|a_n|^p \to 0$.

**Solution:** (a) write $x = p^{\alpha_1}k_1/l_1$ and $y = p^{\alpha_2}k_2/l_2$ ($k_1, k_2, l_1, l_2$ not divisible by $p$). We may assume $\alpha_1 \geq \alpha_2$. Then

$$
|x + y|^p = |p^{\alpha_2}(p^{\alpha_1-\alpha_2}k_1 + l_1k_2)/(l_1l_2)|^p = |p^{\alpha_2}|^p = p^{-\alpha_2}
$$

since $p$ does not divide the terms in parenthesis above. Note that $\max\{|x|^p, |y|^p\} \leq |x|^p + |y|^p$. Hence $d(x, z) = |x - y + y - z|^p \leq |x - y|^p + |y - z|^p$, proving the triangle inequality. Obviously $d$ is symmetric and $d(x, y) = 0$ iff $x = y$.

(b) Consider the partial sums $S_N = \sum_{n=1}^{N} a_n$. Suppose $S_N$ converges. Then $S_N$ is a Cauchy sequence, hence for $\epsilon > 0$ there is $N_\epsilon$ such that

$$
|a_{N_\epsilon}|^p = |S_N - S_{N_\epsilon-1}|^p \leq \epsilon
$$

for $N \geq N_\epsilon$. So $|a_n|^p \to 0$ w.r.t. the metric $d$. Suppose now $a_n \to 0$, i.e. $|a_n|^p \to 0$. Then for $M, N \in \mathbb{N}$

$$
|S_{N+M} - S_M|^p \leq \max\{|a_{N+1}|^p, \ldots, |a_{N+M}|^p\} \to 0
$$

Hence $S_N$ is a Cauchy sequence which converges in the completion of $\mathbb{Q}$ w.r.t. the metric $d$.

**Problem 6:** Let $(X, d)$ be a compact metric space and denote by $B_R(a) \subset X$ the closed ball of radius $R > 0$ centered at $a \in X$. Suppose $\mu$ is a positive Borel measure on $X$ satisfying for some $\beta > 0$ and for all $r \in (0, 1)$ and $a \in X$

$$
c_1 \ r^\beta \leq \mu(B_r(a)) \leq \ r^\beta,
$$

with $c_1 > 0$ independent of $r$ and $a$. Fix a point $a \in X$. Find all $\alpha \in \mathbb{R}$ for which $x \mapsto d(x, a)^\alpha$ is in $L^1(X, d\mu)$.

**Solution:** Only the case $\alpha < 0$ is interesting. Since $d(x, a)^\alpha$ is bounded and continuous away from $a$ it suffices to check whether

$$
\int_{B_1(a)} d(x, a)^\alpha \ d\mu(x) < \infty.
$$

Let $\{R_k\}$ be a strictly monotone decreasing sequence of positive reals and denote by $B_k$ the balls $B_{R_k}(a)$. We will choose $R_k$ such that $B_k \setminus B_{k+1}$ has essentially the same measure as $B_k$. To achieve this we compute

$$
\mu(B_k \setminus B_{k+1}) = \mu(B_k) - \mu(B_{k+1}) \geq c_1 R_k^\beta - R_{k+1}^\beta
$$
Hence, if we choose \( R_{k+1} = R_k \gamma, 0 < \gamma < 1 \), the last term is \( R_k^\beta (c_1 - \gamma^\beta) \) which by appropriate choice of \( \gamma \) equals \( R_k^\beta c_1/2 \). We set \( R_k = \gamma^k, k = 0, 1, \ldots \) and write

\[
\int_{B_{1(a)}} d(x, a)^\alpha \, d\mu(x) = \sum_{k \geq 0} \int_{B_k \setminus B_{k+1}} d(x, a)^\alpha \, d\mu(x)
\]

On each "shell" \( B_k \setminus B_{k+1} \) we may bound the integrand above by \( R_{k+1}^\alpha \) and from below by \( R_k^\alpha \). Since the shells have \( \mu \)-measure at most \( R_k^\beta \) we find that

\[
\int_{B_{1(a)}} d(x, a)^\alpha \, d\mu(x) \leq \sum_k \gamma^{(k+1)\alpha} \gamma^k \beta.
\]

The latter geometric series converges if \( \alpha > -\beta \). Since we also have

\[
\int_{B_k \setminus B_{k+1}} d(x, a)^\alpha \, d\mu(x) \geq \gamma^k \alpha \gamma^k \beta c_1/2.
\]

the condition \( \alpha > -\beta \) is also necessary.

**Problem 7:** Let \( H \) be a Hilbert space. Show that if \( T : H \to H \) is symmetric, i.e. \( \langle x, Ty \rangle = \langle Tx, y \rangle \) for all \( x, y \in H \), then \( T \) is linear and continuous.

**Solution:** First we show that \( T \) is linear. Let \( x_1, x_2 \in H \) then for all \( y \in H \) we have \( \langle T(x_1 + x_2), y \rangle = \langle x_1 + x_2, Ty \rangle = \langle x_1, Ty \rangle + \langle x_2, Ty \rangle = \langle Tx_1, y \rangle + \langle Tx_2, y \rangle = \langle Tx_1 + Tx_2, y \rangle \). Hence \( T(x_1 + x_2) = Tx_1 + Tx_2 \). Similarly one shows that \( T \) is homogeneous. To see that \( T \) is continuous we first note that by the closed graph theorem it suffices to show that the graph \( \text{graph}(T) \) is closed. Let \( (x_n, Tx_n) \in \text{graph}(T) \) be a sequence in \( H \times H \) converging to \( (x, y) \in H \times H \). We claim: \( Tx = y \). To see this consider

\[
\|Tx - y\|^2 = \langle y - Tx, y - Tx \rangle = \lim_n \langle Tx_n - Tx, y - Tx \rangle = \lim_n \langle x_n - x, T(y - Tx) \rangle = 0
\]

Hence \( Tx = y \).