Question 1

(a) Prove that every sequence of real numbers either has a non-decreasing subsequence or a non-increasing subsequence.

(b) Deduce that every bounded sequence of real numbers has a convergent subsequence.

Solution

Let us denote our sequence by $(x_n)_{n=0}^{\infty}$.

(a) We say that the sequence has a *peak* at n_0 if

$$x_{n_0} > x_n \text{ for all } n > n_0.$$

If there are infinitely many peaks, say at $(n_k)_{k=1}^{\infty}$, where

$$n_1 < n_2 < n_3 < ...,$$

then

$$x_{n_1} > x_{n_2} > x_{n_3} > ...$$

so $\left(x_{n_j}\right)_{j=1}^{\infty}$ is a decreasing subsequence. If there are only finitely many peaks, let N be the last peak. Then for all $n > N$, there exists $m > n$ with

$$x_m \geq x_n.$$

(If not, there would be a peak at n, imposible). Then we can construct an increasing subsequence, using induction. To see this, let $n_1 = N$. Since there is not a peak at n_1, there exists $n_2 > n_1$ such that

$$x_{n_2} \geq x_{n_1}.$$

Next, there is not a peak at n_2, so there exists $n_3 > n_2$ such that

$$x_{n_3} \geq x_{n_2}.$$

Assuming that we have chosen $n_1 < n_2 < ... < n_k$, we can choose $n_{k+1} > n_k$ such that

$$x_{n_{k+1}} > x_{n_k},$$

as there is not a peak at n_k. Then $(x_{n_k})_{k=1}^{\infty}$ is increasing.

(b) As $(x_n)_{n=0}^{\infty}$ is bounded, there exists $A > 0$ such that

$$|x_n| \leq A, \, n \geq 0.$$

By (a), $(x_n)_{n=0}^{\infty}$ has either an increasing or decreasing subsequence $\left(x_{n_j}\right)_{j=1}^{\infty}$. If this subsequence is increasing, then it is increasing and bounded above by A, so converges. If this subsequence is decreasing, then it is decreasing and bounded below by $-A$, and so converges.

Question 2

Let (f_n) be a sequence of nonnegative Lebesgue measurable functions on $[0,1]$, and let (E_m) be a sequence of Lebesgue measurable subsets of $[0,1]$.

(a) Suppose that there is an integrable function f such that for $n \geq 1$ and almost every $x \in [0,1]$,

$$f_n(x) \leq f(x). \tag{1}$$

Prove that

$$\limsup_{n\to\infty} \int_0^1 f_n \leq \int_0^1 \limsup_{n\to\infty} f_n. \tag{2}$$

Is the hypothesis (1) for some integrable f necessary?

(b) Let $E_n \subset [0,1]$ for $n \geq 1$ and let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m.$$

(i) Prove that

$$\limsup_{n\to\infty} \; meas\,(E_n) \leq meas\,(E).$$

(ii) Either prove that equality holds in part (i) or give an example in which strict inequality holds.

(iii) Prove that

$$\sum_{k=1}^{\infty} meas\,(E_k) < \infty \;\Rightarrow\; meas\,(E) = 0.$$

Solutions

(a) We apply Fatou's lemma to the non-negative measurable functions $f - f_n$ (They are non-negative a.e. and we can ignore the set of measure 0). We have

$$\liminf_{n\to\infty} \int_0^1 (f - f_n) \geq \int_0^1 \liminf_{n\to\infty} (f - f_n). \tag{1}$$

Since f does not depend on n,

$$\begin{aligned} \liminf_{n\to\infty} (f - f_n) &= f + \liminf_{n\to\infty} (-f_n) \\ &= f - \limsup_{n\to\infty} f_n, \end{aligned}$$

so

$$\int_0^1 \liminf_{n\to\infty} (f - f_n) = \int_0^1 f - \int_0^1 \limsup_{n\to\infty} f_n.$$

Also

$$\begin{aligned} \liminf_{n\to\infty} \left(\int_0^1 f - \int_0^1 f_n \right) &= \int_0^1 f + \liminf_{n\to\infty} \left(-\int_0^1 f_n \right) \\ &= \int_0^1 f - \limsup_{n\to\infty} \int_0^1 f_n. \end{aligned}$$

Plugging these into (1) gives

$$\int_0^1 f - \limsup_{n\to\infty} \int_0^1 f_n \geq \int_0^1 f - \int_0^1 \limsup_{n\to\infty} f_n$$

and hence the result.

The condition (1) for some integrable f is necessary. We can use the same type of counterexamples that are used to show we need a dominating function in Lebesgue's Dominated Convergence Theorem. For example, let

$$f_n(x) = \begin{cases} n^2 x, & x \in \left[0, \frac{1}{n}\right] \\ 0, & x \in \left[\frac{1}{n}, 1\right] \end{cases} .$$

Then we see that if $x \in (0,1]$, we have $f_n(x) = 0$ for $n > \frac{1}{x}$, so

$$\lim_{n\to\infty} f_n(x) = 0.$$

Also $f_n(0) = 0$ for $n \geq 1$. So

$$\lim_{n\to\infty} f_n(x) = 0, x \in [0,1]$$

and then

$$\int_0^1 \limsup_{n\to\infty} f_n = \int_0^1 0 = 0.$$

But

$$\begin{aligned} \int_0^1 f_n &= n^2 \int_0^{1/n} x\, dx \\ &= n^2 \frac{1}{2n^2} = \frac{1}{2}, \end{aligned}$$

so

$$\limsup_{n\to\infty} \int_0^1 f_n = \frac{1}{2} > 0 = \int_0^1 \limsup_{n\to\infty} f_n.$$

(b) (i) Note that $x \in E$ iff $x \in E_n$ for infinitely many n, that is $\chi_{E_n}(x) = 1$ for infinitely many n. So

$$x \in E \Longleftrightarrow \limsup_{n\to\infty} \chi_{E_n}(x) = 1.$$

Hence

$$\chi_E(x) = \limsup_{n\to\infty} \chi_{E_n}(x).$$

Since characteristic functions are bounded above by 1, we can apply (b) to deduce that

$$\begin{aligned} \limsup_{n\to\infty} \int_0^1 \chi_{E_n} &\leq \int_0^1 \limsup_{n\to\infty} \chi_{E_n} \\ &= \int_0^1 \chi_E. \end{aligned}$$

That is,

$$\limsup_{n\to\infty} \ meas\,(E_n) \leq meas\,(E)\,.$$

(ii) No, we don't have equality always. For example, let $E_n = [0, \frac{1}{2})$ if n is odd and $E_n = \left[\frac{1}{2}, 1\right]$ if n is even. Then

$$E = \limsup_{n\to\infty} E_n = [0, 1]$$

as every point of $[0, 1]$ belongs to infinitely many of the $\{E_n\}$, and conversely, each $E_n \subset [0, 1]$. So

$$\int_0^1 \chi_E = \int_0^1 1 = 1.$$

But for each n, E_n has linear measure $\frac{1}{2}$, so

$$\limsup_{n\to\infty} \int_0^1 \chi_{E_n} = \frac{1}{2}.$$

(iii) We have for each n,

$$E \subset \bigcup_{m=n}^{\infty} E_m$$

so

$$meas\,(E) \leq \sum_{m=n}^{\infty} meas\,(E_m)\,.$$

As $n \to \infty$, the right-hand side approaches 0 (because of convergence) and hence

$$meas\,(E) = 0.$$

Question 3

(a) State a necessary and sufficient criterion for a function f on $[0,1]$ to be Riemann integrable. Your criterion must involve Lebesgue measure.

(b) Let $f : [0,1] \to \mathbb{R}$ be Lebesgue integrable. Set $f(x) = f(0)$ for $x < 0$ and $f(x) = f(1)$ for $x > 1$. Prove that

$$\lim_{h \to 0} \int_0^1 |f(x+h) - f(x)|\, dx = 0.$$

(You may assume results about approximation of Lebesgue integrable functions by continuous functions).

(c) Let $g : [0,1] \to [0,1]$ be measurable. Prove that if f is (bounded and) Riemann integrable in $[0,1]$, then

$$\lim_{h \to 0} \int_0^1 |f(x + hg(x)) - f(x)|\, dx = 0.$$

Solution

(a) For f to be Riemann integrable, it is necessary and sufficient that f be continuous a.e.

(b) If first f is continuous in $[0,1]$, then because of the way we extended it, it will be continuous in $[-1,2]$. Then f is uniformly continuous there (a continuous function on a compact interval is unformly continuous). Then given $\varepsilon > 0$, we can find $\delta \in (0,1)$ such that

$$x \in [0,1] \text{ and } |h| < \delta \Rightarrow |f(x+h) - f(x)| < \varepsilon.$$

So

$$|h| < \delta \Rightarrow \int_0^1 |f(x+h) - f(x)|\, dx \leq \int_0^1 \varepsilon\ dx = \varepsilon.$$

Then the result follows for continuous f. Now suppose that we only know f is Lebesgue integrable. Then we can find continuous $g : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 |f - g| < \varepsilon/3.$$

We may also assume that $g(0) = f(0)$ and $g(1) = f(1)$ (just change g in small neighborhoods of $0, 1$ if necessary). Extend g to the real line in the same way we did for f. Then

$$\begin{aligned} & \int_0^1 |f(x+h) - f(x)|\, dx \\ \leq\ & \int_0^1 |f(x+h) - g(x+h)|\, dx + \int_0^1 |g(x+h) - g(x)|\, dx + \int_0^1 |g(x) - f(x)|\, dx \\ <\ & \varepsilon/3 + \int_0^1 |g(x+h) - g(x)|\, dx + \varepsilon/3. \end{aligned}$$

For $|h|$ small enough, as g is continuous, we have

$$\int_0^1 |g(x+h) - g(x)|\, dx < \varepsilon/3$$

and then

$$\int_0^1 |f(x+h) - f(x)|\, dx < \varepsilon.$$

(c) Suppose M is such that

$$|f(x)| \leq M \text{ for all } x.$$

We have

$$|f(x + hg(x)) - f(x)| \leq 2M \text{ for all } x \in [0,1].$$

Moreover, as g is bounded above by 1 and below by 0, we have at each point of continuity of f,

$$\lim_{h \to 0} f(x + hg(x)) = f(x).$$

Hence a.e. (recall f is Riemann integrable),

$$\lim_{h \to 0} |f(x + hg(x)) - f(x)| = 0.$$

By Lebesgue's Dominated Convergence Theorem,

$$\lim_{h \to 0} \int_0^1 |f(x + hg(x)) - f(x)|\, dx = 0.$$

Question 4

Let $1 < p < \infty$ and $\{f_n\}$ be a sequence of functions in $L_p[0,1]$. Give a proof of, or counterexample to, the following assertions:
(a) If $f_n \to f$ weakly in $L_p[0,1]$ as $n \to \infty$, then there is a subsequence $\{f_{n_k}\}$ that converges a.e. to f.
(b) If $f_n \to f$ in norm in $L_p[0,1]$ as $n \to \infty$, then there is a subsequence $\{f_{n_k}\}$ that converges a.e. to f.

Solutions

(a) This is false. Let φ be the periodic function

$$\varphi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1 \end{cases}.$$

For $n \geq 1$, let

$$f_n(t) = \varphi(2^n t), \, t \in [0,1].$$

Then

$$\lim_{n \to \infty} \int_0^1 f_n \chi_A = 0$$

for every measurable subset A of $[0,1]$. Here χ_A denotes the characteristic function of A. (We can first see this for dyadic intervals $\left[\frac{j}{2^\ell}, \frac{k}{2^\ell}\right]$, and then for general intervals, then for finite unions of intervals, and by dominated convergence for any measurable set). It follows that $f_n \to 0$ weakly as $n \to \infty$, but no subsequence converges a.e., since $|f_n(t)| = 1$ for all n and t.

(b) This is true. Let $\varepsilon > 0$ and *meas* denote linear Lebesgue measure. For $n \geq 1$,

$$\begin{aligned} & \varepsilon \, meas(\{t : |f_n - f|(t) \geq \varepsilon\}) \\ \leq & \int_{\{t:|f_n - f|(t) \geq \varepsilon\}} |f_n - f|^p \\ \leq & \int_0^1 |f_n - f|^p \to 0, \, n \to \infty. \end{aligned}$$

It follows that for each $\varepsilon > 0$,

$$meas(\{t : |f_n - f|(t) \geq \varepsilon\}) \to 0 \text{ as } n \to \infty.$$

That is, $f_n \to f$ in measure. By a standard argument, there is a subsequence of $\{f_n\}$ that converges a.e. to f.

Outline of this argument:

Choose $n_1 < n_2 < n_3 < ...$ such that for $k \geq 1$,

$$E_k = \left\{ t : |f_{n_k} - f|(t) \geq \frac{1}{2^k} \right\}$$

has

$$meas(E_k) \leq \frac{1}{2^k}.$$

Let

$$E = \limsup_{k\to\infty} E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k.$$

Then it is easy to check that $meas\,(E) = 0$ and for $t \notin E$,

$$\lim_{k\to\infty} f_{n_k}\,(t) = f\,(t)\,.$$

Question 5

Let f be a function defined on $\mathbb{R}^2$ with continuous second partial derivatives. Use Fubini's theorem to give an easy proof that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(Hint: Assume it is not true, and integrate $\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}$ over a suitable square).

Solution

Let us suppose the result is false. Then at some point

$$\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \neq 0.$$

Let us suppose that it is positive at that point. By continuity, we may find a square $S = [a, a+h] \times [b, b+h]$ containing that point, in which

$$\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} > 0.$$

Then we know

$$I = \int \int_S \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right] dx\ dy > 0. \tag{1}$$

Here

$$\begin{aligned} I &= \int \int_S \frac{\partial^2 f}{\partial x \partial y} dx\ dy - \int \int_S \frac{\partial^2 f}{\partial y \partial x} dx\ dy \\ &= I_1 - I_2. \end{aligned} \tag{2}$$

Because we are dealing with a continuous integrand, we may write these integrals as iterated integrals, and may change the order of integration, and may also use the fundamental theorem of calculus:

$$\begin{aligned} I_1 &= \int_b^{b+h} \left(\int_a^{a+h} \frac{\partial^2 f}{\partial x \partial y} dx \right) dy \\ &= \int_b^{b+h} \left(\frac{\partial f}{\partial y} (a+h, y) - \frac{\partial f}{\partial y} (a, y) \right) dy \\ &= f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a, b). \end{aligned}$$

Next,

$$\begin{aligned} I_2 &= \int_a^{a+h} \left(\int_b^{b+h} \frac{\partial^2 f}{\partial y \partial x} dy \right) dx \\ &= \int_a^{a+h} \left(\frac{\partial f}{\partial x} (x, b+h) - \frac{\partial f}{\partial x} (x, b) \right) dx \\ &= f(a+h, b+h) - f(a, b+h) - f(a+h, b) + f(a, b). \end{aligned}$$

Thus $I_1 = I_2$ and (2) gives $I = 0$, contradicting (1).

Question 6

(a) Show that there is a bounded linear functional φ on ℓ_∞ such that

$$\varphi(\mathbf{x}) = \lim_{i\to\infty} x_i$$

for every convergent sequence $\mathbf{x} = (x_i)$ in ℓ_∞.

(b) Is this linear functional unique?

Solution

(a) Let c denote the set of all convergent sequences in ℓ_∞. It is a closed subspace of ℓ_∞. Note that φ is a linear functional on c. This follows from the linearity of limits. Moreover, φ is a bounded/continuous linear functional. Indeed if $\mathbf{x} = (x_i) \in c$, then

$$\begin{aligned} |\varphi(\mathbf{x})| &= \left|\lim_{i\to\infty} x_i\right| = \lim_{i\to\infty} |x_i| \\ &\leq \sup_i |x_i| = \|\mathbf{x}\|. \end{aligned}$$

So φ has norm at most one. By the Hahn-Banach theorem, φ has a bounded/continuous extension to all of ℓ_∞.

(b) It is not unique. Let c_e denote the set of all sequences in ℓ_∞ whose even index components converge. Thus $\mathbf{x} = (x_i) \in c_e$ iff

$$\lim_{i\to\infty} x_{2i} \text{ exists.}$$

Similarly, let c_o denote the set of all sequences in ℓ_∞ whose odd index components converge. Thus $\mathbf{x} = (x_i) \in c_e$ iff

$$\lim_{i\to\infty} x_{2i+1} \text{ exists.}$$

Suppose we first extend φ above to c_e by

$$\varphi(\mathbf{x}) = \lim_{i\to\infty} x_{2i}.$$

It clearly is a bounded linear extension. Now we extend via Hahn-Banach to all of ℓ_∞. Call the resulting extension φ_1. Next, extend our original φ from c to c_o by

$$\varphi(\mathbf{x}) = \lim_{i\to\infty} x_{2i+1}.$$

Now we extend via Hahn-Banach to all of ℓ_∞. Call the resulting extension φ_2.

Both φ_1 and φ_2 are bounded linear functional extending φ but they are not equal. To see this, let $\mathbf{x}$ denote the sequence with

$$x_{2i} = 0; x_{2i+1} = 1$$

for all i. We have $\mathbf{x} \in c_e \cap c_0$ and

$$\begin{aligned} \varphi_1(\mathbf{x}) &= \lim_{i\to\infty} x_{2i} = 0; \\ \varphi_2(\mathbf{x}) &= \lim_{i\to\infty} x_{2i+1} = 1. \end{aligned}$$

Question 7

Prove that there is no norm under which the vector space P of polynomials with real coefficients is a Banach space.

Solution

There is no norm under which P is complete. Let us assume there is, and derive a contradiction. We claim that

$$P_n = \{\text{polynomials } P \text{ of degree } \leq n\}$$

is a closed subspace of P, and is also a nowhere dense subset of P.

Indeed, suppose (p_k) is a sequence of polynomials of degree $\leq n$ with $\|p_k - p\| \to 0$, $k \to \infty$, for some function p. Now P_n is a finite dimensional subspace of a Banach space, so $K = \{q \in P_n : \|p - q\| \leq 1\}$, which is closed and bounded,is also compact. As $p_k \in K$ for large enough k, we deduce that $p \in P_n$ also. Thus P_n is closed.

To see that it is nowhere dense, we must show it has empty interior. But that is obvious: if $\ell > n$ and ε is small enough, while $p \in P_n$, then

$$q(x) = \varepsilon x^{\ell} + p(x) \notin P_n$$

but

$$\|q - p\| = \varepsilon \left\|x^{\ell}\right\|$$

may be made as small as we please. So P_n must have empty interior. Then

$$P = \bigcup_{n=1}^{\infty} P_n$$

is a countable union of nowhere dense sets, so is of the first category. This contradicts Baire's theorem for closed metric spaces (and hence Banach spaces.)