# Analysis Comprehensive Exam Questions
Fall 2007

1. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces, and fix $1 < p < \infty$. Show that if $f$ is a measurable function on $X \times Y$, then

$$\left(\int_Y \left(\int_X |f(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \le \int_X \left(\int_Y |f(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x). \tag{1}$$

<u>Solution</u>

Let $p'$ be the dual index to $p$. Define

$$F(y) = \int_X |f(x,y)|\, d\mu(x).$$

Then the left-hand side of equation (1) can be rewritten as:

$$\left(\int_Y \left(\int_X |f(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} = \left(\int_Y |F(y)|^p\, d\nu(y)\right)^{1/p} = \|F\|_p.$$

We estimate this as follows:

$$\begin{aligned}
\|F\|_p^p &= \int_Y F(y)^{p-1}\, F(y)\, d\nu(y) \\
&= \int_Y F(y)^{p-1} \int_X |f(x,y)|\, d\mu(x)\, d\nu(y) \\
&= \int_X \int_Y F(y)^{p-1}\, |f(x,y)|\, d\nu(y)\, d\mu(x) \qquad \text{(Tonelli)} \\
&\le \int_X \left(\int_Y F(y)^{(p-1)p'}\, d\nu(y)\right)^{1/p'} \left(\int_Y |f(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x) \qquad \text{(Hölder)} \\
&= \int_X \left(\int_Y F(y)^p\, d\nu(y)\right)^{1/p'} \left(\int_Y |f(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x) \\
&= \|F\|_p^{p-1} \int_X \left(\int_Y |f(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x).
\end{aligned}$$

Dividing through by $\|F\|_p^{p-1}$, we therefore obtain

$$\|F\|_p \le \int_X \left(\int_Y |f(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x),$$

which is equation (1). $\square$

2. Let $(X, M, \mu)$ be a measure space, let $\mu$ be a positive measure, and let $f$, $f_n \in L^1(X, M, \mu)$ for $1 \le n < \infty$. Assume that:

(1) $f_n(x) \to f(x)$ for almost every $x \in X$,

(2) $\|f_n\|_1 \to \|f\|_1$.

Prove $\|f_n - f\|_1 \to 0$.

Solution

Define $h_n = (|f| + |f_n|) - |f - f_n|$, which is nonnegative. Then by Fatou's lemma,

$$\begin{aligned}
\int 2|f|\,d\mu &= \int \liminf_{n\to\infty} h_n\,d\mu \\
&\le \liminf_{n\to\infty} \int h_n\,d\mu \\
&= \int |f|\,d\mu + \liminf_{n\to\infty}\left(\int |f_n|\,d\mu - \int |f - f_n|\,d\mu\right) \\
&\le \int |f|\,d\mu + \limsup_{n\to\infty}\int |f_n|\,d\mu + \liminf_{n\to\infty}\left(-\int |f - f_n|\,d\mu\right) \\
&= 2\int |f|\,d\mu - \limsup_{n\to\infty}\left(\int |f - f_n|\,d\mu\right)
\end{aligned}$$

Since $\int |f|\,d\mu$ is finite, one can subtract it from both sides to get

$$\limsup_{n\to\infty} \int_A |f - f_n|\,d\mu \le 0,$$

and hence $\|f - f_n\|_1 \to 0$. □

3. Let $X$ be a Banach space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a Schauder basis for $X$ if for each $x \in X$ there exist unique scalars $a_n(x)$ such that

$$x = \sum_{n=1}^{\infty} a_n(x)\, x_n,$$

where the series converges in the norm of $X$. It can be shown (you may take this as given) that $a_n \in X^*$ for each $n$.

Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for a Banach space $X$ and $\{y_n\}_{n\in\mathbb{N}}$ is a Schauder basis for a Banach space $Y$. Prove that the following two statements are equivalent.

(a) There exists a continuous linear bijection $S\colon X \to Y$ such that $S(x_n) = y_n$ for each $n \in \mathbb{N}$.

(b) Given scalars $c_n$,

$$\sum_{n=1}^{\infty} c_n x_n \text{ converges in } X \quad\Longleftrightarrow\quad \sum_{n=1}^{\infty} c_n y_n \text{ converges in } Y.$$

Solution

(a) $\Rightarrow$ (b). Suppose that statement (a) holds, and that $x = \sum c_n x_n$ converges in $X$. Then since $S$ is linear and continuous, we have that $S(x) = \sum c_n S(x_n) = \sum c_n y_n$ converges in $X$. To see why exactly this is true, note that $x = \sum c_n x_n$ means that

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} c_n x_n \right\| = 0.$$

Therefore,

$$\begin{aligned} \left\| S(x) - \sum_{n=1}^{N} c_n y_n \right\| &= \left\| S(x) - \sum_{n=1}^{N} c_n S(x_n) \right\| \\ &= \left\| S\Big( x - \sum_{n=1}^{N} c_n x_n \Big) \right\| \\ &\le \|S\| \left\| x - \sum_{n=1}^{N} c_n x_n \right\| \to 0, \end{aligned}$$

so $\sum c_n y_n$ converges in $Y$ to $S(x)$.

The Inverse Mapping Theorem tells us that $S^{-1}$ is continuous, so a symmetric argument using $S^{-1}$ shows that if $\sum c_n y_n$ converges in $Y$, then $\sum c_n x_n$ converges in $X$.

(b) $\Rightarrow$ (a). Suppose that (b) holds. By definition of Schauder basis, there exist functionals $a_n \in X^*$ such that

$$x = \sum_{n=1}^{\infty} a_n(x)\, x_n, \qquad x \in X,$$

and there exist functionals $b_n \in Y^*$ that satisfy

$$y = \sum_{n=1}^{\infty} b_n(y)\, y_n, \qquad y \in Y.$$

Choose any $x \in X$. Then $x = \sum a_n(x)\, x_n$ converges in $X$, so by hypothesis

$$S(x) = \sum_{n=1}^{\infty} a_n(x)\, y_n$$

converges in $Y$. $S$ defined in this way is linear, and we will show that it is a continuous bijection of $X$ onto $Y$.

Suppose that $S(x) = 0$. Then we have

$$\sum_{n=1}^{\infty} a_n(x)\, y_n = S(x) = 0 = \sum_{n=1}^{\infty} 0\, y_n.$$

The uniqueness of the coefficients therefore implies that $a_n(x) = 0$ for every $n$, and hence $x = \sum a_n(x)\, x_n = 0$. Therefore $S$ is injective.

Next, if $y$ is any element of $Y$, then $y = \sum b_n(y)\, y_n$ converges in $Y$, so by hypothesis $x = \sum b_n(y)\, x_n$ converges in $X$. The uniqueness of the coefficients implies that $b_n(y) = a_n(x)$ for every $n$. Hence $S(x) = y$ and therefore $S$ is surjective. Thus $S$ is a bijection of $X$ onto $Y$.

Now we show that $S$ is continuous. For each $N$, define $S_N \colon X \to Y$ by

$$S_N(x) = \sum_{n=1}^{N} a_n(x)\, y_n.$$

Since each functional $a_n$ is continuous, we conclude that each $S_N$ is continuous. And since $S_N(x) \to S(x)$, the Banach–Steinhaus Theorem implies that $S$ is continuous, which completes the proof.

Alternatively, we can appeal directly to the Uniform Boundedness Principle (of which the Banach–Steinhaus Theorem is simply a special case). We have that $S_N(x) \to S(x)$, so

$$\forall\, x \in X, \quad \sup_N \|S_N(x)\| < \infty.$$

Since each $S_N$ is bounded, the Uniform Boundedness Principle implies that Hence

$$\|S(x)\| \le \limsup_{N\to\infty} \|S_N\|\, \|x\| \le M\, \|x\|,$$

so $S$ is bounded. $\square$

4. Prove that if $f$ is integrable on $[a, b]$ and

$$\int_a^x f(t)\, dt = 0 \tag{2}$$

for all $x \in [a, b]$, then $f(t) = 0$ a.e. in $[a, b]$.

Solution

Without loss of generality, we may suppose $f(x) > 0$ on some set $E$ of positive measure (a similar argument applies if $f(x)$ is negative on a a set of positive measure). Because $|E| > 0$, then there exists a closed set $F \subset E$ with $|F| > 0$. Let $O = [a, b] \setminus F$. Since

$$0 = \int_a^b f(t)\, dt = \int_F f(t)\, dt + \int_O f(t)\, dt,$$

we have

$$\int_O f(t)\, dt = -\int_F f(t)\, dt \neq 0.$$

Since $O$ is open, it is a union of disjoint open intervals, say,

$$O = \bigcup_n (a_n, b_n).$$

Then

$$\int_O f(t)\, dt = \sum_n \int_{a_n}^{b_n} f(t)\, dt \neq 0,$$

so there must be an $n$ such that

$$\int_{a_n}^{b_n} f(t)\, dt \neq 0.$$

But then either

$$\int_a^{a_n} f(t)\, dt \neq 0 \qquad \text{or} \qquad \int_a^{b_n} f(t)\, dt \neq 0,$$

which contradicts the condition (2).

An alternative approach is to use the Lebesgue Differentiation Theorem. □

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and assume that $\mu$ is a bounded measure, i.e., $\mu(X) < \infty$. Fix $1 \le p < \infty$, and assume that $F \in L^p(X)'$, the dual space of $L^p(X)$. Show that there exists a $g \in L^1(X)$ such that

$$\forall\, A \in \mathcal{M}, \quad F(\chi_A) = \int_A g(x)\, d\mu(x).$$

Notes: You cannot assume that $L^p(X)' \cong L^{p'}(X)$; this problem is one step in the proof of that isomorphism. You may assume that the scalar field is $\mathbb{R}$, so that all linear functionals are real-valued.

<u>Solution</u>

We are given that $F$ is a bounded linear functional on $L^p(X)$. Define $\lambda\colon \Sigma \to \mathbb{R}$ by

$$\lambda(A) = F(\chi_A), \quad A \in \mathcal{M}.$$

We claim that $\lambda$ is a signed measure on $X$.

First, $\lambda(\emptyset) = F(0) = 0$.

Second, to show that $\lambda$ is countably additive, suppose that $E_k$, $k \in \mathbb{N}$, are disjoint measurable subsets of $X$. Define

$$A = \bigcup_{k=1}^{\infty} E_k, \qquad A_N = \bigcup_{k=1}^{N} E_k, \ N \in \mathbb{N}.$$

Then $\mu(A_N) \to \mu(A)$ by continuity from above. On the other hand, since $\mu$ is a bounded measure, we have that $\mu(A \backslash A_N) = \mu(A) - \mu(A_N)$, and hence $\mu(A \backslash A_N) \to 0$. Hence

$$\|\chi_A - \chi_{A_N}\|_p^p = \int_X |\chi_A(x) - \chi_{A_N}(x)|^p\, dx = \int_X |\chi_{A \backslash A_N}(x)|^p\, dx = \mu(A \backslash A_N) \to 0.$$

Hence $\chi_{A_N} \to \chi_A$ in $L^p(X)$. But $F$ is a continuous linear functional on $L^p(X)$, so this implies that $F(\chi_{A_N}) \to F(\chi_A)$. Hence, because the $E_k$ are disjoint, we have

$$\begin{aligned}
\lambda(A) = F(A) &= \lim_{N\to\infty} F(\chi_{A_N}) \\
&= \lim_{N\to\infty} F\left(\sum_{k=1}^{N} \chi_{E_k}\right) \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} F(\chi_{E_k}) \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} \lambda(E_k) \\
&= \sum_{k=1}^{\infty} \lambda(E_k).
\end{aligned}$$

Therefore $\lambda$ is countably additive and hence is a signed measure on $X$.

Now, if $E \in \mathcal{M}$ and $\mu(A) = 0$, then we have $\chi_A = 0$ $\mu$-a.e., and hence $\lambda(A) = F(\chi_A) = F(0) = 0$. Therefore $\lambda$ is absolutely continuous with respect to $\mu$, i.e., $\lambda \ll \mu$. The Radon–Nikodym theorem therefore implies that there exists a $g \in L^1(X)$ such that

$$F(\chi_A) = \lambda(A) = \int_X g(x)\, d\mu(x), \quad A \in \mathcal{M}. \quad \square$$

6. (a) Suppose $\phi$ is a real function on $\mathbb{R}$ such that

$$\phi\left(\int_0^1 f(x)\,dx\right) \le \int_0^1 \phi(f(x))\,dx, \tag{3}$$

for every real bounded measurable function $f$. Prove that $\phi$ is convex.

(b) Let $\phi$ be a convex function on $\mathbb{R}$. Prove that the inequality (3) holds for each integrable function $f$ on $[0,1]$.

<u>Solution</u>

Given any two finite real values $a$ and $b$ and given an arbitrary $\lambda \in [0,1]$, define

$$f(x) = \begin{cases} a, & \lambda < x \le 1, \\ b, & 0 \le x \le \lambda. \end{cases}$$

Clearly, $f(x)$ is a real bounded measurable function. Further,

$$\begin{aligned} \phi(\int_0^1 f(x)\,dx) &= \phi\left(\int_0^\lambda f(x)\,dx + \int_\lambda^1 f(x)\,dx\right) \\ &= \phi\left(\int_0^\lambda b\,dx + \int_\lambda^1 a\,dx\right) \\ &= \phi(\lambda b + (1-\lambda)a). \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} \int_0^1 \phi(f(x))\,dx &= \int_0^\lambda \phi(f(x))\,dx + \int_\lambda^1 \phi(f(x))\,dx \\ &= \int_0^\lambda \phi(b)\,dx + \int_\lambda^1 \phi(a)\,dx \\ &= \lambda\phi(b) + (1-\lambda)\phi(a). \end{aligned} \tag{5}$$

Putting (4) and (5) back into (3), we obtain

$$\phi(\lambda b + (1-\lambda)a) \le \lambda\phi(b) + (1-\lambda)\phi(a),$$

which confirms that $\phi$ is convex.

(b) This part is Jensen's inequality. Let $\alpha = \int_0^1 f(t)\,dt$, and let $y = m(x-\alpha)+\phi(\alpha)$ be the equation of a supporting line at $\alpha$, where $m$ is taken to lie between the left- and right-hand derivatives of $\phi$ at $\alpha$. Since the supporting line always lies below the graph of $\phi$, we have

$$m(x-\alpha) + \phi(\alpha) \le \phi(x).$$

Replacing $x$ by $f(t)$, we obtain for almost every $t \in (0,1)$ that

$$m(f(t)-\alpha) + \phi(\alpha) \le \phi(f(t)).$$

Integrating both sides with respect to $t$ then gives equation 3. $\square$

7. Let $f$ be a bounded linear functional on a separable Hilbert space $H$. Prove that there is a unique $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x$ and, moreover, $\|f\| = \|y\|$.

Solution

The result is also true for arbitrary Hilbert spaces, but since we have assumed that $H$ is separable, we can use the fact that there exists a complete orthonomal system (basis) $\{\phi_\nu\}_{\nu\in\mathbb{N}}$ for $H$.

Set $b_\nu = f(\phi_\nu)$. Then for each finite $n$, we have

$$\sum_{\nu=1}^{n} b_\nu^2 = f\left(\sum_{\nu=1}^{n} b_\nu\phi_\nu\right) \le \|f\| \left\|\sum_{\nu=1}^{n} b_\nu\phi_\nu\right\| \le \|f\| \left(\sum_{\nu=1}^{n} b_\nu^2\right)^{1/2}.$$

This implies that

$$\sum_{\nu=1}^{n} b_\nu^2 \le \|f\|^2, \qquad \text{all } n,$$

and therefore

$$\sum_{\nu=1}^{\infty} b_\nu^2 \le \|f\|^2 < \infty.$$

Hence the series

$$y = \sum_{\nu=1}^{\infty} b_\nu\phi_\nu,$$

converges, and furthermore

$$\|y\|^2 = \sum_{\nu=1}^{\infty} b_\nu^2 \le \|f\|^2.$$

Given any $x \in H$, set

$$a_\nu = \langle x, \phi_\nu\rangle.$$

Then

$$x = \sum_{\nu=1}^{\infty} a_\nu\phi_\nu.$$

Since $\sum_{\nu=1}^{n} a_\nu\phi_\nu \to x$, we have by the continuity and linearity of $f$ that

$$f(x) = \lim_{n\to\infty} f\left(\sum_{\nu=1}^{n} a_\nu\phi_\nu\right) = \lim_{n\to\infty}\sum_{\nu=1}^{n} a_\nu b_\nu = \sum_{\nu=1}^{\infty} a_\nu b_\nu = \langle x, y\rangle.$$

Finally, by the Schwarz inequality, we have

$$\|f\| \le \|y\|,$$

so $\|f\| = \|y\|$. $\square$

8. Let $(X, \mathcal{M}, \mu)$ and be a measure space such that $\mu(X) < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions, and let $f$ be a measurable function. Prove that

$$f_n \to f \text{ in measure} \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \int_X \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu = 0.$$

<u>Solution</u>

$\Rightarrow$. Suppose that $f_n \to f$ in measure, and choose any $\varepsilon > 0$. Then

$$\begin{aligned}
\int_X \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu &= \int_{|f-f_n|>\varepsilon} \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu + \int_{|f-f_n|\le\varepsilon} \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu \\
&\le \int_{|f-f_n|>\varepsilon} 1\, d\mu + \int_{|f-f_n|\le\varepsilon} \frac{\varepsilon}{1}\, d\mu \\
&\le \mu\{|f - f_n| > \varepsilon\} + \varepsilon\, \mu(X).
\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} \int_X \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu \le \limsup_{n\to\infty} \left(\mu\{|f - f_n| > \varepsilon\} + \varepsilon\, \mu(X)\right) = \varepsilon \mu(X).$$

Since $\mu(X) < \infty$ and $\varepsilon$ is arbitrary, we conclude that $\lim_{n\to\infty} \int_X \frac{|f-f_n|}{1+|f-f_n|}\, d\mu = 0$.

$\Leftarrow$. Assume that $\lim_{n\to\infty} \int_X \frac{|f-f_n|}{1+|f-f_n|}\, d\mu = 0$. Choose any $\varepsilon > 0$. Note that

$$x \ge \varepsilon \qquad \Longrightarrow \qquad \frac{x}{1+x} \ge \frac{\varepsilon}{1+\varepsilon},$$

so

$$\begin{aligned}
\mu\{|f - f_n| > \varepsilon\} d\mu &= \frac{1+\varepsilon}{\varepsilon} \int_{|f-f_n|>\varepsilon} \frac{\varepsilon}{1+\varepsilon}\, d\mu \\
&\le \frac{1+\varepsilon}{\varepsilon} \int_{|f-f_n|>\varepsilon} \frac{|f - f_n|}{1 + |f - f_n|}\, d\mu \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence $f_n \to f$ in measure. □