Analysis Comprehensive Exam Questions
Fall 2008

1. (a) Let $E \subseteq \mathbb{R}$ be measurable with finite Lebesgue measure $|E|$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(E)$ and there exists a function f such that $f_n(x) \to f(x)$ for a.e. $x \in E$. Show that $\|f - f_n\|_1 \to 0$ as $n \to \infty$.

(b) Show that the conclusion of part (a) can fail if $|E| = \infty$.

Solution

(a) Choose $\varepsilon > 0$, and let $C = \sup_n \|f_n\|_2 < \infty$. By Fatou's Lemma, we have

$$\|f\|_2^2 \;=\; \int_E \lim_{n\to\infty} |f_n|^2 \;\le\; \liminf_{n\to\infty} \int_E |f_n|^2 \;=\; \liminf_{n\to\infty} \|f_n\|_2^2 \;\le\; C.$$

Hence $f \in L^2(E)$.

By Egorov's Theorem, there exists $A \subseteq E$ such that

$$|E\backslash A| < \left(\frac{\varepsilon}{4C}\right)^2$$

and $f_n \to f$ uniformly on A. Therefore, we can find an N such that

$$\|(f - f_n)\,\chi_A\|_\infty \;<\; \frac{\varepsilon}{2|E|}, \qquad \text{all } n > N.$$

Then for $n > N$ we have by Cauchy–Schwarz that

$$\begin{aligned}
\|f - f_n\|_1 \;&=\; \int_A |f - f_n| + \int_{E\backslash A} |f - f_n| \\
&\le\; |A|\;\|(f - f_n)\,\chi_A\|_\infty + \left(\int_{E\backslash A} |f - f_n|^2\right)^{1/2} \left(\int_{E\backslash A} 1\right)^{1/2} \\
&<\; |A|\,\frac{\varepsilon}{2|E|} + \|f - f_n\|_2\;|E\backslash A|^{1/2} \\
&<\; \frac{\varepsilon}{2} + 2C\,\frac{\varepsilon}{4C} \;=\; \varepsilon. \qquad \square
\end{aligned}$$

(b) Let $f_n = \chi_{[n,n+1]}$. Then $\|f_n\|_2 = 1$ for every n, and $f_n(x) \to 0$ for every x. However, f_n does not converge to the zero function in L^1-norm, since $\|f_n\|_1 = 1$. $\square$

2. Let X be a Banach space and let T, S be bounded linear operators on X. Prove that:

(a) $I - TS$ has a bounded inverse if and only if $I - ST$ has a bounded inverse.

(b) $\sigma(TS)\backslash\{0\} = \sigma(ST)\backslash\{0\}$.

Remark: $\sigma(A)$ denotes the spectrum of A.

Solution

(a) Suppose that $I - TS$ has a bounded inverse. In particular, $I - TS$ is injective. Suppose that $(I - ST)v = 0$ for some $v \in X$. Then we have $T(I - ST)v = (I - TS)Tv = 0$, so $Tv = 0$. But this implies that $v = (I - ST)v = 0$. Hence also $I - ST$ is injective.

On the other hand since $I - TS$ is surjective we have that for every $z \in X$ there exists an $x \in X$ such that $(I - TS)x = Tz$. Observe that this implies that $x \in T(X)$ since $x = T(Sx + z) = Ty$. We thus have that $T(I - ST)y = Tz$, or $(I - ST)y = z + v$ with $v \in \mathrm{Ker}(T)$. But then, setting $w = y - v$, we have that $(I - ST)w = z$ and $I - ST$ is surjective.

Thus $I - ST$ is a bounded bijection of X onto itself, and therefore has a bounded inverse by the Open Mapping Theorem.

(b) Suppose $\lambda \notin \sigma(TS)$ and $\lambda \neq 0$. Then $TS - \lambda I$ has a bounded inverse, so $I - \frac{T}{\lambda}S$ has a bounded inverse. By part (a) it follows that $I - S\frac{T}{\lambda}$ and thus $ST - \lambda I$ has a bounded inverse, so $\lambda \notin \sigma(ST)$. $\square$

3. Let f, g be absolutely continuous functions on $[0,1]$. Show that for $x \in [0,1]$ we have

$$\int_0^x f(t)\, g'(t)\, dt \;=\; f(x)g(x) - f(0)g(0) - \int_0^x f'(t)\, g(t)\, dt.$$

Solution

Since f, g are absolutely continuous, we know that they are differentiable almost everywhere and that f', $g' \in L^1[0,1]$. Consequently, $f'(s)\, g'(t) \in L^1\big([0,1]^2\big)$. Letting $E = \{(s,t) \in [0,x]^2 : s \le t\}$, we compute that

$$\begin{aligned}
\iint_E f'(s)\, g'(t)\, ds\, dt \;&=\; \int_0^x \left(\int_0^t f'(s)\, ds \right) g'(t)\, dt \\
&=\; \int_0^x \big(f(t) - f(0)\big)\, g'(t)\, dt \\
&=\; \int_0^x f(t)\, g'(t)\, dt - f(0) \int_0^x g'(t)\, dt \\
&=\; \int_0^x f(t)\, g'(t)\, dt - f(0)\, \big(g(x) - g(0)\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\iint_E f'(s)\, g'(t)\, dt\, ds \;&=\; \int_0^x f'(s) \left(\int_s^x g'(t)\, dt \right) ds \\
&=\; \int_0^x f'(s)\, \big(g(x) - g(s)\big)\, dt \\
&=\; g(x) \int_0^x f'(s)\, ds - \int_0^x f'(s)\, g(s)\, ds \\
&=\; g(x)\big(f(x) - f(0)\big) - \int_0^x f'(s)\, g(s)\, ds.
\end{aligned}$$

Finally, Fubini's Theorem implies that these two integrals are equal, so the result follows. $\square$

4. Let $f : [0,1] \to \mathbb{R}$ a bounded function whose set of discontinuities D is closed and nowhere dense.

(a) Is it true that every such f is Riemann integrable?

(b) Prove that for every such f there exists an homeomorphism $h : [0,1] \to [0,1]$ such that $f \circ h$ is Riemann integrable.

Remark: A homeomorphism is a continuous bijection that has a continuous inverse.

Solution

(a) Clearly no. Let $\{q_1, q_2, \ldots, q_n, \ldots\}$ be an ordering of the rational numbers in $(0,1)$ and set

$$I \;=\; \bigcup_n B(q_n, \varepsilon 2^{-n}),$$

where $B(x,r) = (x - r/2, x + r/2) \cap (0,1)$. Thus $|I| \leq \varepsilon$ but I is open and dense. Thus $J = [0,1] \backslash I$ is closed and nowhere dense but with large positive measure. Observe that $f = \chi_J$ is continuous for every $x \in I$ since I is open, but it is discontinuous for every $x \in J$ since I is dense. Hence f is discontinuous on a closed nowhere dense set of positive measure and thus it is not Riemann integrable.

(b) Let D be the set of discontinuities of f and $D^c = [0,1] \backslash D$. We can define

$$g(x) \;=\; \frac{1}{1 - |D|} \int_0^x \chi_{D^c}(t)\, dt.$$

Observe that $g(0) = 0$, $g(1) = 1$, and g is continuous and strictly increasing. Indeed, if $x < y$, there exists an open interval $I \subset (x,y)$ such that $I \subset D^c$ since D is closed and nowhere dense. From this we have that

$$g(y) - g(x) \;=\; \frac{1}{1 - |D|} \int_x^y \chi_{D^c}(t)\, dt \;\geq\; |I| \;>\; 0.$$

Thus g is an invertible function and its inverse is continuous. Finally since D is closed we have that D^c is the union of countably many open interval I_i. Observe that

$$|g(I_i)| \;=\; \frac{1}{1 - |D|} \int_{I_i} \chi_{D^c}(t)\, dt \;=\; \frac{1}{1 - |D|}\, |I_i|,$$

so $|g(D^c)| = 1$ and $|g(D)| = 0$. Hence we can choose $h = g^{-1}$, for then $f \circ h$ is discontinuous on the set $g(D)$, which has measure zero, and therefore $f \circ g$ is Riemann integrable. $\square$

5. Let X be a Banach space with norm $\|\cdot\|_X$. Assume that Y is proper subspace of X that is dense in X with respect to $\|\cdot\|_X$, and that there is another norm $\|\cdot\|_Y$ on Y with respect to which Y is a Banach space. Show that if there exists a constant C such that

$$\|x\|_X \;\le\; C\,\|x\|_Y \quad \text{for all } x \in Y,$$

then there exists a continuous linear functional on $(Y, \|\cdot\|_Y)$ that has no extension to a continuous linear functional on $(X, \|\cdot\|_X)$.

Solution

The hypotheses imply that Y is continuously embedded into X, i.e., if $i\colon Y \to X$ is given by $i(x) = x$ for $x \in Y$ then i is continuous and $\|i\| \le C$. The adjoint of i is the restriction map $R\colon X^* \to Y^*$ given by $R(\mu) = \mu|_Y$. Hence R is bounded, with $\|R\| \le C$. That is, $\|\mu|_Y\|_{Y^*} \le C\,\|\mu\|_{X^*}$ for each $\mu \in X^*$. This can also be proved without recourse to adjoints by observing that if $x \in Y$ and $\mu \in X^*$ then

$$|\langle x, \mu|_Y\rangle| \;=\; |\langle x, \mu\rangle| \;\le\; \|\mu\|_{X^*}\,\|x\|_X \;\le\; C\,\|\mu\|_{X^*}\,\|x\|_Y,$$

so $\|\mu|_Y\|_{Y^*} \le C\,\|\mu\|_{X^*}$ (we are using the linear functional notation $\langle x, \mu\rangle = \mu(x)$).

Suppose now that every continuous linear functional on $(Y, \|\cdot\|_Y)$ had an extension to a continuous linear functional on $(X, \|\cdot\|_X)$. Then R is onto. Further, if $\mu \in X^*$ and $R(\mu) = \mu|_Y = 0$, then $\mu = 0$ since μ is continuous and Y is dense in X. Therefore R is injective. Thus $R\colon Y^* \to X^*$ is a bounded bijection, so the Inverse Mapping Theorem implies that R^{-1} is bounded. Combining this with the above facts, there exist $c, C > 0$ such that

$$\forall\, \mu \in X^*, \qquad c\,\|\mu\|_{X^*} \le \|\mu|_Y\|_{Y^*} \le C\,\|\mu\|_{X^*}.$$

Now fix any $x \in Y$. Then by Hahn–Banach, there exists a $\nu \in Y^*$ such that $\|\nu\|_{Y^*} = 1$ and $|\langle x, \nu\rangle| = \|x\|_Y$. By hypothesis, there exists an extension of ν to a continuous linear functional on $(X, \|\cdot\|_X)$. Call this extension μ, so we have $\mu|_Y = \nu$. Then

$$\begin{aligned}
\|x\|_Y \;=\; |\langle x, \nu\rangle| \;&=\; |\langle x, \mu\rangle| \\
&\le\; \|x\|_X\,\|\mu\|_{X^*} \\
&\le\; \|x\|_X\,\frac{1}{c}\,\|\mu|_Y\|_{Y^*} \\
&=\; \|x\|_X\,\frac{1}{c}\,\|\nu\|_{Y^*} \\
&=\; \frac{1}{c}\,\|x\|_X.
\end{aligned}$$

Since we also have $\|x\|_X \le C\,\|x\|_Y$, we conclude that $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent norms on Y. But Y is complete with respect to $\|\cdot\|_Y$, and therefore it is complete with respect to $\|\cdot\|_X$. Consequently, Y is closed with respect to $\|\cdot\|_X$. However, Y is dense in X with respect to $\|\cdot\|_X$, which implies that $Y = X$, a contradiction. $\square$

6. Let G be an unbounded open subset of $\mathbb{R}$. Prove that

$$H \;=\; \{x \in \mathbb{R} : kx \in G \text{ for infinitely many } k \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$.

Solution

If kx belongs G for infinitely many k then, for every $n > 0$, x belongs to

$$\bigcup_{|k|>n} G/k$$

where

$$G/k \;=\; \{y \in \mathbb{R} : ky \in G\}.$$

Vice versa, if $x \in \bigcup_{|k|>n} G/k$ for every $n > 0$, then $kx \in G$ for infinitely many k. Thus

$$H \;=\; \bigcap_{n=1}^{\infty} \bigcup_{|k|>n} G/k.$$

Clearly $\cup_{|k|>n} G/k$ is an open set. By the Baire Category Theorem, it is therefore enough to prove that $\cup_{k>n} G/k$ is dense, for then H must be dense.

Let $D = (z_-, z_+)$ be any open interval. If

$$D \cap \bigcup_{|k|>n} G/k \;=\; \emptyset,$$

then

$$\bigcup_{|k|>n} kD \cap G \;=\; \emptyset.$$

Without loss of generality, assume that $z_- > 0$. Then for k large enough we have that $(k+1)z_- > kz_+$, and hence $\bigcup_{k>n} kD$ contains a subset of the form (d, ∞). By considering negative k we likewise conclude that $\bigcup_{k>n} kD$ contains $(-\infty, -d)$. Consequently, G cannot contain $(-\infty, -d) \cup (d, \infty)$, which contradicts the fact that G is unbounded. $\square$

7. Let μ_1, μ_2 be bounded signed Borel measures on $\mathbb{R}$. Show that there exists a unique bounded signed Borel measure μ such that

$$\int f\,d\mu \;=\; \int\Big(\int f(x+y)\,d\mu_1(x)\Big)\,d\mu_2(y), \qquad f \in C_c(\mathbb{R}).$$

Show further that $\|\mu\| \le \|\mu_1\|\,\|\mu_2\|$.

Note: Scalars in this problem are real.

<u>Solution</u>

If E is any Borel set in $\mathbb{R}$, then

$$\iint \chi_E(x+y)\,d|\mu_1|(x)\,d|\mu_2|(y) \;\le\; \iint d|\mu_1|(x)\,d|\mu_2|(y) \;=\; \|\mu_1\|\,\|\mu_2\| \;<\; \infty.$$

Hence, by Fubini's Theorem, we can define

$$\mu(E) \;=\; \iint \chi_E(x+y)\,d\mu_1(x)\,d\mu_2(y),$$

and we have $|\mu(E)| \le \|\mu_1\|\,\|\mu_2\|$.

We claim that μ defined in this way is a signed Borel measure. The above work shows that $\mu(E)$ is a finite real number for every Borel set E, and we clearly have that $\mu(\emptyset) = 0$. Hence we need only show that μ is countably additive.

Suppose that $E_1, E_2, \ldots$ are disjoint Borel sets, and let $E = \cup E_j$. For each x and y, we have that

$$0 \;\le\; \sum_{j=1}^{N} \chi_{E_j}(x+y) \;\to\; \chi_E(x+y) \;\le\; 1 \;\in\; L^1(\mu_1 \times \mu_2).$$

Therefore, by the Dominated Convergence Theorem,

$$\begin{aligned}
\mu(E) \;&=\; \iint \chi_E(x+y)\,d\mu_1(x)\,d\mu_2(y)\\
&=\; \lim_{j\to\infty} \iint \sum_{j=1}^{N} \chi_{E_j}(x+y)\,d\mu_1(x)\,d\mu_2(y)\\
&=\; \lim_{j\to\infty} \sum_{j=1}^{N} \iint \chi_{E_j}(x+y)\,d\mu_1(x)\,d\mu_2(y)\\
&=\; \lim_{j\to\infty} \sum_{j=1}^{N} \mu(E_j)\\
&=\; \sum_{j=1}^{\infty} \mu(E_j).
\end{aligned}$$

Therefore μ is a signed Borel measure.

If we let $\mathbb{R} = P \cup N$ be a Hahn decomposition of $\mathbb{R}$ for μ, then

$$\begin{aligned}
\|\mu\| \;=\; |\mu|(\mathbb{R}) \;&=\; \mu(P) - \mu(N) \\
&= \iint \chi_P(x+y)\,d\mu_1(x)\,d\mu_2(y) \;-\; \iint \chi_N(x+y)\,d\mu_1(x)\,d\mu_2(y) \\
&\le \iint \chi_P(x+y)\,d|\mu_1|(x)\,d|\mu_2|(y) \;+\; \iint \chi_N(x+y)\,d|\mu_1|(x)\,d\mu_2|(y) \\
&= \iint d|\mu_1|(x)\,d|\mu_2|(y) \;=\; \|\mu_1\|\,\|\mu_2\|.
\end{aligned}$$

If $\phi = \sum_{k=1}^n a_k\,\chi_{E_k}$ is any simple function, then

$$\begin{aligned}
\int \phi\,d\mu \;=\; \sum_{k=1}^n a_k \int \chi_{E_k}\,d\mu \;&=\; \sum_{k=1}^n a_k \iint \chi_{E_k}(x+y)\,d\mu_1(x)\,d\mu_2(y) \\
&= \iint \phi(x+y)\,d\mu_1(x)\,d\mu_2(y).
\end{aligned}$$

If we fix $f \in C_c(\mathbb{R})$, then there exist simple functions ϕ_k such that $|\phi_k| \le |f|$ and $\phi_k \to f$ pointwise. Since $f \in L^1(\mu)$ and $f(x+y) \in L^1(\mu_1 \times \mu_2)$, we therefore have by the Dominated Convergence Theorem that

$$\begin{aligned}
\iint f(x+y)\,d\mu_1(x)\,d\mu_2(y) \;&=\; \lim_{k\to\infty} \iint \phi_k(x+y)\,d\mu_1(x)\,d\mu_2(y) \\
&= \lim_{k\to\infty} \int \phi_k\,d\mu \;=\; \int f\,d\mu.
\end{aligned}$$

It remains only to show that μ is unique. If ν is another signed Borel measure that satisfies

$$\int f\,d\nu \;=\; \int\left(\int f(x+y)\,d\mu_1(x)\right) d\mu_2(y), \qquad f \in C_c(\mathbb{R}), \tag{1}$$

then we have $\int f\,d(\mu-\nu) = 0$ for every $f \in C_c(\mathbb{R})$. By the Riesz Representation Theorem, $C_c(\mathbb{R})^* = M_b(\mathbb{R})$, the space of finite signed Borel measures on $\mathbb{R}$. Therefore we must have $\mu = \nu$.

As the Riesz Representation Theorem for $C_c(X)$ is not part of the Comprehensive Exam syllabus, we give an alternative direct proof. As above, suppose that ν is another signed Borel measure that satisfies equation (1). Fix any open interval (a,b). Let $f_n \in C_c(\mathbb{R})$ be such that $0 \le f_n \le 1$ and $f_n \to \chi_{(a,b)}$ pointwise. Then by the Dominated Convergence Theorem, we have

$$\mu(a,b) \;=\; \lim_{n\to\infty} \int f_n\,d\mu \;=\; \lim_{n\to\infty} \int f_n\,d\nu \;=\; \nu(a,b).$$

This extends from open intervals to all Borel sets, so we conclude that $\mu = \nu$. $\square$

8. Given $1 \le p < \infty$ and $f_n \in L^p(\mathbb{R})$, prove that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R})$ if and only if the following three conditions hold ($|E|$ denotes Lebesgue measure).

(a) $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure.

(b) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|E| < \delta$ then $\int_E |f_n|^p < \varepsilon$ for every n.

(c) For every $\varepsilon > 0$ there exists a set E with $|E| < \infty$ such that $\int_{E^C} |f_n|^p < \varepsilon$ for every n.

Solution

$\Rightarrow$. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^p(\mathbb{R})$. Since $L^p(\mathbb{R})$ is complete, there exists a function $f_0 \in L^p(\mathbb{R})$ such that $f_n \to f_0$ in L^p-norm.

(a) By Tchebyshev's inequality,

$$|\{|f_m - f_n| \ge \varepsilon\}| \;\le\; \frac{1}{\varepsilon^p}\,\|f_m - f_n\|_p^p,$$

so $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure.

(b) Given $\varepsilon > 0$, we have by standard arguments that for each $n \ge 0$ there exists a $\delta_n > 0$ such that if $|E| < \delta_n$ then $\int_E |f_n|^p < \varepsilon$. Since $f_n \to f_0$, there exists an N such that $\|f_n - f_0\|_p < \varepsilon$ for all $n \ge N$. Set

$$\delta \;=\; \min\{\delta_0, \delta_1, \dots, \delta_N\},$$

and suppose that $|E| < \delta$. Then we have $\int_E |f_n|^p \le \varepsilon$ for $n \le N$, and if $n > N$ then

$$\left(\int_E |f_n|^p\right)^{1/p} \;\le\; \left(\int_E |f_n - f_0|^p\right)^{1/p} + \left(\int_E |f_0|^p\right)^{1/p} \;\le\; \|f_n - f_0\|_p + \varepsilon \;<\; 2\varepsilon.$$

Hence statement (b) holds.

(c) Choose $\varepsilon > 0$. Since for each $f \in L^p(\mathbb{R})$ we have $\int_{|x|>m} |f|^p \to 0$ as $m \to \infty$, for each $n \ge 0$ we can find a set E_n with $|E_n| < \infty$ such that

$$\int_{E_n^C} |f_n|^p \;<\; \varepsilon^p, \quad \text{all } n \ge 0.$$

Let $E = E_0 \cup E_1 \cup \cdots \cup E_N$, where N is such that $\|f_n - f_0\|_p < \varepsilon$ for all $n \ge N$. Then $|E| < \infty$, and if $n > N$ then

$$\left(\int_{E^C} |f_n|^p\right)^{1/p} \;\le\; \left(\int_{E^C} |f_0 - f_n|^p\right)^{1/p} + \left(\int_{E^C} |f_0|^p\right)^{1/p} \;\le\; \|f_0 - f_n\|_p + \varepsilon \;\le\; 2\varepsilon.$$

Since $E_1, \dots, E_N \subseteq E_0$, we also have the required inequality for $n \le N$, so statement (c) holds.

$\Leftarrow$. Assume statements (a)–(c) hold and choose $\varepsilon > 0$. Let the set E be given as in statement (c). Set

$$A_{mn} \;=\; \left\{|f_m - f_n| \ge \left(\frac{\varepsilon}{|E|}\right)^{1/p}\right\}.$$

Let δ be as given in statement (b). By statement (a), there exists an N such that $|A_{mn}| < \delta$ for all m, $n \geq N$. Hence

$$\begin{aligned}
\|f_m - f_n\|_p^p &\leq \int_{A_{mn}} |f_m - f_n|^p + \int_{E\setminus A_{mn}} |f_m - f_n|^p + \int_{E^{\mathrm{C}}} |f_m - f_n|^p \\
&\leq \int_{A_{mn}} 2^p \left(|f_m|^p + |f_n|^p\right) + \int_{E\setminus A_{mn}} \frac{\varepsilon}{|E|} + \int_{E^{\mathrm{C}}} 2^p \left(|f_m|^p + |f_n|^p\right) \\
&\leq 2^{p+1}\,\varepsilon + \varepsilon + 2^{p+1}\,\varepsilon.
\end{aligned}$$

Hence $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^p(\mathbb{R})$. $\square$