1. (a) Let $E \subseteq \mathbb{R}$ be measurable with finite Lebesgue measure $|E|$. Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(E)$ and there exists a function $f$ such that $f_n(x) \to f(x)$ for a.e. $x \in E$. Show that $\|f - f_n\|_1 \to 0$ as $n \to \infty$.

(b) Show that the conclusion of part (a) can fail if $|E| = \infty$.

Solution

(a) Choose $\varepsilon > 0$, and let $C = \sup_n \|f_n\|_2 < \infty$. By Fatou’s Lemma, we have
\[
\|f\|_2^2 = \int_E \lim_{n \to \infty} |f_n|^2 \leq \liminf_{n \to \infty} \int_E |f_n|^2 = \liminf_{n \to \infty} \|f_n\|_2^2 \leq C.
\]
Hence $f \in L^2(E)$.

By Egorov’s Theorem, there exists $A \subseteq E$ such that
\[
|E \setminus A| < \left(\frac{\varepsilon}{4C}\right)^2
\]
and $f_n \to f$ uniformly on $A$. Therefore, we can find an $N$ such that
\[
\|(f - f_n) \chi_A\|_\infty < \frac{\varepsilon}{2|E|}, \quad \text{all } n > N.
\]
Then for $n > N$ we have by Cauchy–Schwarz that
\[
\|f - f_n\|_1 = \int_A |f - f_n| + \int_{E \setminus A} |f - f_n|
\]
\[
\leq |A| \|(f - f_n) \chi_A\|_\infty + \left(\int_{E \setminus A} |f - f_n|^2\right)^{1/2} \left(\int_{E \setminus A} 1\right)^{1/2}
\]
\[
< |A| \frac{\varepsilon}{2|E|} + \|f - f_n\|_2 |E \setminus A|^{1/2}
\]
\[
< \frac{\varepsilon}{2} + 2C \frac{\varepsilon}{4C} = \varepsilon. \quad \Box
\]

(b) Let $f_n = \chi_{[n,n+1]}$. Then $\|f_n\|_2 = 1$ for every $n$, and $f_n(x) \to 0$ for every $x$. However, $f_n$ does not converge to the zero function in $L^1$-norm, since $\|f_n\|_1 = 1$. \[\]
2. Let $X$ be a Banach space and let $T, S$ be bounded linear operators on $X$. Prove that:

(a) $I - TS$ has a bounded inverse if and only if $I - ST$ has a bounded inverse.

(b) $\sigma(TS) \setminus \{0\} = \sigma(ST) \setminus \{0\}$.

Remark: $\sigma(A)$ denotes the spectrum of $A$.

Solution

(a) Suppose that $I - TS$ has a bounded inverse. In particular, $I - TS$ is injective. Suppose that $(I - ST)v = 0$ for some $v \in X$. Then we have $T(I - ST)v = (I - TS)Tv = 0$, so $Tv = 0$. But this implies that $v = (I - ST)v = 0$. Hence also $I - ST$ is injective.

On the other hand since $I - TS$ is surjective we have that for every $z \in X$ there exists an $x \in X$ such that $(I - TS)x = Tz$. Observe that this implies that $x \in T(X)$ since $x = T(Sx + z) = Ty$. We thus have that $T(I - ST)y = Tz$, or $(I - ST)y = z + v$ with $v \in \text{Ker}(T)$. But then, setting $w = y - v$, we have that $(I - ST)w = z$ and $I - ST$ is surjective.

Thus $I - ST$ is a bounded bijection of $X$ onto itself, and therefore has a bounded inverse by the Open Mapping Theorem.

(b) Suppose $\lambda \notin \sigma(TS)$ and $\lambda \neq 0$. Then $TS - \lambda I$ has a bounded inverse, so $I - \frac{T}{\lambda}S$ has a bounded inverse. By part (a) it follows that $I - \frac{T}{\lambda}X$ and thus $ST - \lambda I$ has a bounded inverse, so $\lambda \notin \sigma(ST)$. \qed
3. Let $f$, $g$ be absolutely continuous functions on $[0, 1]$. Show that for $x \in [0, 1]$ we have
\[
\int_0^x f(t) g'(t) \, dt = f(x)g(x) - f(0)g(0) - \int_0^x f'(t) g(t) \, dt.
\]

**Solution**

Since $f$, $g$ are absolutely continuous, we know that they are differentiable almost everywhere and that $f'$, $g' \in L^1[0, 1]$. Consequently, $f'(s) g'(t) \in L^1([0, 1]^2)$. Letting $E = \{(s, t) \in [0, x]^2 : s \leq t\}$, we compute that
\[
\int \int_E f'(s) g'(t) \, ds \, dt = \int_0^x \left( \int_0^t f'(s) \, ds \right) g'(t) \, dt
\]
\[
= \int_0^x (f(t) - f(0)) g'(t) \, dt
\]
\[
= \int_0^x f(t) g'(t) \, dt - f(0) \int_0^x g'(t) \, dt
\]
\[
= \int_0^x f(t) g'(t) \, dt - f(0) (g(x) - g(0)).
\]

On the other hand,
\[
\int \int_E f'(s) g'(t) \, dt \, ds = \int_0^x f'(s) \left( \int_s^x g'(t) \, dt \right) \, ds
\]
\[
= \int_0^x f'(s) (g(x) - g(s)) \, ds
\]
\[
= g(x) \int_0^x f'(s) \, ds - \int_0^x f'(s) g(s) \, ds
\]
\[
= g(x) \left( f(x) - f(0) \right) - \int_0^x f'(s) g(s) \, ds.
\]

Finally, Fubini’s Theorem implies that these two integrals are equal, so the result follows. □
4. Let \( f : [0, 1] \to \mathbb{R} \) a bounded function whose set of discontinuities \( D \) is closed and nowhere dense.

(a) Is it true that every such \( f \) is Riemann integrable?

(b) Prove that for every such \( f \) there exists an homeomorphism \( h : [0, 1] \to [0, 1] \) such that \( f \circ h \) is Riemann integrable.

Remark: A homeomorphism is a continuous bijection that has a continuous inverse.

Solution

(a) Clearly no. Let \( \{q_1, q_2, \ldots, q_n, \ldots\} \) be an ordering of the rational numbers in \((0, 1)\) and set
\[
I = \bigcup_n B(q_n, \varepsilon 2^{-n}),
\]
where \( B(x, r) = (x - r/2, x + r/2) \cap (0, 1) \). Thus \(|I| \leq \varepsilon \) but \( I \) is open and dense. Thus \( J = [0, 1] \setminus I \) is closed and nowhere dense but with large positive measure. Observe that \( f = \chi_J \) is continuous for every \( x \in I \) since \( I \) is open, but it is discontinuous for every \( x \in J \) since \( I \) is dense. Hence \( f \) is discontinuous on a closed nowhere dense set of positive measure and thus it is not Riemann integrable.

(b) Let \( D \) be the set of discontinuities of \( f \) and \( D^c = [0, 1] \setminus D \). We can define
\[
g(x) = \frac{1}{1 - |D|} \int_0^x \chi_{D^c}(t) \, dt.
\]
Observe that \( g(0) = 0 \), \( g(1) = 1 \), and \( g \) is continuous and strictly increasing. Indeed, if \( x < y \), there exists an open interval \( I \subset (x, y) \) such that \( I \subset D^c \) since \( D \) is closed and nowhere dense. From this we have that
\[
g(y) - g(x) = \frac{1}{1 - |D|} \int_x^y \chi_{D^c}(t) \, dt \geq |I| > 0.
\]
Thus \( g \) is an invertible function and its inverse is continuous. Finally since \( D \) is closed we have that \( D^c \) is the union of countably many open interval \( I_i \). Observe that
\[
|g(I_i)| = \frac{1}{1 - |D|} \int_{I_i} \chi_{D^c}(t) \, dt = \frac{1}{1 - |D|} |I_i|,
\]
so \( |g(D^c)| = 1 \) and \( |g(D)| = 0 \). Hence we can choose \( h = g^{-1} \), for then \( f \circ h \) is discontinuous on the set \( g(D) \), which has measure zero, and therefore \( f \circ g \) is Riemann integrable. \( \square \)
5. Let $X$ be a Banach space with norm $\| \cdot \|_X$. Assume that $Y$ is proper subspace of $X$ that is dense in $X$ with respect to $\| \cdot \|_X$, and that there is another norm $\| \cdot \|_Y$ on $Y$ with respect to which $Y$ is a Banach space. Show that if there exists a constant $C$ such that

$$\|x\|_X \leq C \|x\|_Y$$

for all $x \in Y$, then there exists a continuous linear functional on $(Y, \| \cdot \|_Y)$ that has no extension to a continuous linear functional on $(X, \| \cdot \|_X)$.

**Solution**

The hypotheses imply that $Y$ is continuously embedded into $X$, i.e., if $i: Y \to X$ is given by $i(x) = x$ for $x \in Y$ then $i$ is continuous and $\|i\| \leq C$. The adjoint of $i$ is the restriction map $R: X^* \to Y^*$ given by $R(\mu) = \mu|_Y$. Hence $R$ is bounded, with $\|R\| \leq C$. That is, $\|\mu|_Y\|_{Y^*} \leq C \|\mu\|_{X^*}$ for each $\mu \in X^*$. This can also be proved without recourse to adjoints by observing that if $x \in Y$ and $\mu \in X^*$ then

$$|\langle x, \mu \rangle| = |\langle x, \mu \rangle| \leq \|\mu\|_{X^*} \|x\|_X \leq C \|\mu\|_{X^*} \|x\|_Y,$$

so $\|\mu|_Y\|_{Y^*} \leq C \|\mu\|_{X^*}$ (we are using the linear functional notation $\langle x, \mu \rangle = \mu(x)$).

Suppose now that every continuous linear functional on $(Y, \| \cdot \|_Y)$ had an extension to a continuous linear functional on $(X, \| \cdot \|_X)$. Then $R$ is onto. Further, if $\mu \in X^*$ and $R(\mu) = \mu|_Y = 0$, then $\mu = 0$ since $\mu$ is continuous and $Y$ is dense in $X$. Therefore $R$ is injective. Thus $R: X^* \to X^*$ is a bounded bijection, so the Inverse Mapping Theorem implies that $R^{-1}$ is bounded. Combining this with the above facts, there exist $c, C > 0$ such that

$$\forall \mu \in X^*, \quad c \|\mu\|_{X^*} \leq \|\mu|_Y\|_{Y^*} \leq C \|\mu\|_{X^*}.$$

Now fix any $x \in Y$. Then by Hahn–Banach, there exists a $\nu \in Y^*$ such that $\|\nu\|_{Y^*} = 1$ and $|\langle x, \nu \rangle| = \|x\|_Y$. By hypothesis, there exists an extension of $\nu$ to a continuous linear functional on $(X, \| \cdot \|_X)$. Call this extension $\mu$, so we have $\mu|_Y = \nu$. Then

$$\|x\|_Y = |\langle x, \nu \rangle| = |\langle x, \mu \rangle|$$

$$\leq \|x\|_X \|\mu\|_{X^*}$$

$$\leq \|x\|_X \frac{1}{c} \|\mu|_Y\|_{Y^*}$$

$$= \|x\|_X \frac{1}{c} \|\nu\|_{Y^*}$$

$$= \frac{1}{c} \|x\|_X.$$

Since we also have $\|x\|_X \leq C \|x\|_Y$, we conclude that $\| \cdot \|_X$ and $\| \cdot \|_Y$ are equivalent norms on $Y$. But $Y$ is complete with respect to $\| \cdot \|_Y$, and therefore it is complete with respect to $\| \cdot \|_X$. Consequently, $Y$ is closed with respect to $\| \cdot \|_X$. However, $Y$ is dense in $X$ with respect to $\| \cdot \|_X$, which implies that $Y = X$, a contradiction. □
6. Let $G$ be an unbounded open subset of $\mathbb{R}$. Prove that
\[ H = \{ x \in \mathbb{R} : kx \in G \text{ for infinitely many } k \in \mathbb{Z} \} \]
is dense in $\mathbb{R}$.

Solution
If $kx$ belongs $G$ for infinitely many $k$ then, for every $n > 0$, $x$ belongs to
\[ \bigcup_{|k| > n} G/k \]
where
\[ G/k = \{ y \in \mathbb{R} : ky \in G \} \].
Vice versa, if $x \in \bigcup_{|k| > n} G/k$ for every $n > 0$, then $kx \in G$ for infinitely many $k$. Thus
\[ H = \bigcap_{n=1}^{\infty} \bigcup_{|k| > n} G/k. \]

Clearly $\bigcup_{|k| > n} G/k$ is an open set. By the Baire Category Theorem, it is therefore enough to prove that $\bigcup_{k > n} G/k$ is dense, for then $H$ must be dense.

Let $D = (z_-, z_+)$ be any open interval. If $D \cap \bigcup_{|k| > n} G/k = \emptyset$, then $\bigcup_{|k| > n} kD \cap G = \emptyset$.

Without loss of generality, assume that $z_- > 0$. Then for $k$ large enough we have that $(k + 1)z_- > kz_+$, and hence $\bigcup_{k > n} kD$ contains a subset of the form $(d, \infty)$. By considering negative $k$ we likewise conclude that $\bigcup_{k > n} kD$ contains $(-\infty, -d)$. Consequently, $G$ cannot contain $(-\infty, -d) \cup (d, \infty)$, which contradicts the fact that $G$ is unbounded. \qed
7. Let $\mu_1, \mu_2$ be bounded signed Borel measures on $\mathbb{R}$. Show that there exists a unique bounded signed Borel measure $\mu$ such that

$$\int f \, d\mu = \int \left( \int f(x+y) \, d\mu_1(x) \right) \, d\mu_2(y), \quad f \in C_c(\mathbb{R}).$$

Show further that $\|\mu\| \leq \|\mu_1\| \|\mu_2\|$.

Note: Scalars in this problem are real.

Solution

If $E$ is any Borel set in $\mathbb{R}$, then

$$\iint \chi_E(x+y) \, d|\mu_1|(x) \, d|\mu_2|(y) \leq \iint d|\mu_1|(x) \, d|\mu_2|(y) = \|\mu_1\| \|\mu_2\| < \infty.$$ 

Hence, by Fubini’s Theorem, we can define

$$\mu(E) = \iint \chi_E(x+y) \, d\mu_1(x) \, d\mu_2(y),$$

and we have $|\mu(E)| \leq \|\mu_1\| \|\mu_2\|$.

We claim that $\mu$ defined in this way is a signed Borel measure. The above work shows that $\mu(E)$ is a finite real number for every Borel set $E$, and we clearly have that $\mu(\emptyset) = 0$. Hence we need only show that $\mu$ is countably additive.

Suppose that $E_1, E_2, \ldots$ are disjoint Borel sets, and let $E = \bigcup E_j$. For each $x$ and $y$, we have that

$$0 \leq \sum_{j=1}^N \chi_{E_j}(x+y) \rightarrow \chi_E(x+y) \leq 1 \in L^1(\mu_1 \times \mu_2).$$

Therefore, by the Dominated Convergence Theorem,

$$\mu(E) = \iint \chi_E(x+y) \, d\mu_1(x) \, d\mu_2(y)$$

$$= \lim_{j \to \infty} \iint \sum_{j=1}^N \chi_{E_j}(x+y) \, d\mu_1(x) \, d\mu_2(y)$$

$$= \lim_{j \to \infty} \sum_{j=1}^N \iint \chi_{E_j}(x+y) \, d\mu_1(x) \, d\mu_2(y)$$

$$= \lim_{j \to \infty} \sum_{j=1}^N \mu(E_j)$$

$$= \sum_{j=1}^{\infty} \mu(E_j).$$

Therefore $\mu$ is a signed Borel measure.
If we let \( \mathbb{R} = P \cup N \) be a Hahn decomposition of \( \mathbb{R} \) for \( \mu \), then
\[
\|\mu\| = |\mu|(\mathbb{R}) = \mu(P) - \mu(N)
\]
\[
= \int \int \chi_P(x+y) \, d\mu_1(x) \, d\mu_2(y) - \int \int \chi_N(x+y) \, d\mu_1(x) \, d\mu_2(y)
\]
\[
\leq \int \int \chi_P(x+y) \, d|\mu_1|(x) \, d|\mu_2|(y) + \int \int \chi_N(x+y) \, d|\mu_1|(x) \, d|\mu_2|(y)
\]
\[
= \int d|\mu_1|(x) \, d|\mu_2|(y) = \|\mu_1\| \|\mu_2\|.
\]

If \( \phi = \sum_{k=1}^n a_k \chi_{E_k} \) is any simple function, then
\[
\int \phi \, d\mu = \sum_{k=1}^n a_k \int \chi_{E_k} \, d\mu = \sum_{k=1}^n a_k \int \chi_{E_k}(x+y) \, d\mu_1(x) \, d\mu_2(y)
\]
\[
= \int \int \phi(x+y) \, d\mu_1(x) \, d\mu_2(y).
\]

If we fix \( f \in C_c(\mathbb{R}) \), then there exist simple functions \( \phi_k \) such that \( |\phi_k| \leq |f| \) and \( \phi_k \to f \) pointwise. Since \( f \in L^1(\mu) \) and \( f(x+y) \in L^1(\mu_1 \times \mu_2) \), we therefore have by the Dominated Convergence Theorem that
\[
\int \int f(x+y) \, d\mu_1(x) \, d\mu_2(y) = \lim_{k \to \infty} \int \phi_k(x+y) \, d\mu_1(x) \, d\mu_2(y)
\]
\[
= \lim_{k \to \infty} \int \phi_k \, d\mu = \int f \, d\mu.
\]

It remains only to show that \( \mu \) is unique. If \( \nu \) is another signed Borel measure that satisfies equation (1), then we have \( \int f \, d(\mu - \nu) = 0 \) for every \( f \in C_c(\mathbb{R}) \). By the Riesz Representation Theorem, \( C_c(\mathbb{R})^* = M_b(\mathbb{R}) \), the space of finite signed Borel measures on \( \mathbb{R} \). Therefore we must have \( \mu = \nu \).

As the Riesz Representation Theorem for \( C_c(X) \) is not part of the Comprehensive Exam syllabus, we give an alternative direct proof. As above, suppose that \( \nu \) is another signed Borel measure that satisfies equation (1). Fix any open interval \((a, b)\). Let \( f_n \in C_c(\mathbb{R}) \) be such that \( 0 \leq f_n \leq 1 \) and \( f_n \to \chi_{(a,b)} \) pointwise. Then by the Dominated Convergence Theorem, we have
\[
\mu(a,b) = \lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int f_n \, d\nu = \nu(a,b).
\]
This extends from open intervals to all Borel sets, so we conclude that \( \mu = \nu \). \( \square \)
8. Given $1 \leq p < \infty$ and $f_n \in L^p(\mathbb{R})$, prove that $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R})$ if and only if the following three conditions hold ($|E|$ denotes Lebesgue measure).

(a) $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in measure.

(b) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|E| < \delta$ then $\int_E |f_n|^p < \varepsilon$ for every $n$.

(c) For every $\varepsilon > 0$ there exists a set $E$ with $|E| < \infty$ such that $\int_{E^c} |f_n|^p < \varepsilon$ for every $n$.

Solution

$\Rightarrow$. Assume that $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in $L^p(\mathbb{R})$. Since $L^p(\mathbb{R})$ is complete, there exists a function $f_0 \in L^p(\mathbb{R})$ such that $f_n \to f_0$ in $L^p$-norm.

(a) By Tchebyshev’s inequality,

$$| \{|f_m - f_n| \geq \varepsilon\} | \leq \frac{1}{\varepsilon^p} \|f_m - f_n\|_p^p,$$

so $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in measure.

(b) Given $\varepsilon > 0$, we have by standard arguments that for each $n \geq 0$ there exists a $\delta_n > 0$ such that if $|E| < \delta_n$ then $\int_E |f_n|^p < \varepsilon$. Since $f_n \to f_0$, there exists an $N$ such that $\|f_n - f_0\|_p < \varepsilon$ for all $n \geq N$. Set

$$\delta = \min\{\delta_0, \delta_1, \ldots, \delta_N\},$$

and suppose that $|E| < \delta$. Then we have $\int_E |f_n|^p \leq \varepsilon$ for $n \leq N$, and if $n > N$ then

$$\left(\int_E |f_n|^p\right)^{1/p} \leq \left(\int_E |f_n - f_0|^p\right)^{1/p} + \left(\int_E |f_0|^p\right)^{1/p} \leq \|f_n - f_0\|_p + \varepsilon < 2\varepsilon.$$

Hence statement (b) holds.

(c) Choose $\varepsilon > 0$. Since for each $f \in L^p(\mathbb{R})$ we have $\int_{|x| > m} |f|^p \to 0$ as $m \to \infty$, for each $n \geq 0$ we can find a set $E_n$ with $|E_n| < \infty$ such that

$$\int_{E_n^c} |f_n|^p < \varepsilon^p, \quad \text{all } n \geq 0.$$

Let $E = E_0 \cup E_1 \cup \cdots \cup E_N$, where $N$ is such that $\|f_n - f_0\|_p < \varepsilon$ for all $n \geq N$. Then $|E| < \infty$, and if $n > N$ then

$$\left(\int_{E^c} |f_n|^p\right)^{1/p} \leq \left(\int_{E^c} |f_0 - f_n|^p\right)^{1/p} + \left(\int_{E^c} |f_0|^p\right)^{1/p} \leq \|f_0 - f_n\|_p + \varepsilon \leq 2\varepsilon.$$

Since $E_1, \ldots, E_N \subseteq E_0$, we also have the required inequality for $n \leq N$, so statement (c) holds.

$\Leftarrow$. Assume statements (a)–(c) hold and choose $\varepsilon > 0$. Let the set $E$ be given as in statement (c). Set

$$A_{mn} = \left\{ |f_m - f_n| \geq \left(\frac{\varepsilon}{|E|}\right)^{1/p} \right\}.$$
Let $\delta$ be as given in statement (b). By statement (a), there exists an $N$ such that $|A_{mn}| < \delta$ for all $m, n \geq N$. Hence

$$
\|f_m - f_n\|_p^p \leq \int_{A_{mn}} |f_m - f_n|^p + \int_{E \setminus A_{mn}} |f_m - f_n|^p + \int_{E_C} |f_m - f_n|^p
$$

$$
\leq \int_{A_{mn}} 2^p (|f_m|^p + |f_n|^p) + \int_{E \setminus A_{mn}} \frac{\varepsilon}{|E|} + \int_{E_C} 2^p (|f_m|^p + |f_n|^p)
$$

$$
\leq 2^{p+1} \varepsilon + \varepsilon + 2^{p+1} \varepsilon.
$$

Hence $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in $L^p(\mathbb{R})$. \qed