Problem 1. The *closed linear span* of a subset $\{y_j\}$ of a normed vector space X is defined as the intersection of all closed subspaces containing all y_j (and thus the smallest such subspace).

(1) Show that the closed linear span of $\{y_j\}$ is the closure of the *linear span* Y of $\{y_j\}$, which consists of all finite linear combinations of the y_j.

(2) Prove that a point z of a normed linear space X belongs to the closed linear span S of a subset $\{y_j\}$ of X if and only if every bounded linear functional ℓ that vanishes on the subset vanishes at z. That is, if and only if

$$\ell(y_j) = 0 \quad \text{for all } y_j \tag{1}$$

implies that $\ell(z) = 0$.

Hint: For (2), use the Hahn-Banach theorem.

Proof.

(1) Since S contains all y_j, we have that $Y \subset S$. Since S is closed, we obtain $\bar{Y} \subset S$.

To prove the converse, note that the closure $\bar{Y}$ of the subspace Y is again a subspace. So $\bar{Y}$ is a closed subspace containing all y_j. But S is precisely the intersection of all such subspaces, and therefore $S \subset \bar{Y}$.

(2) Since ℓ is linear, assumption (1) implies that $\ell(y) = 0$ for all $y \in Y$. For all $z \in S$ there exists a sequence of $y^n \in Y$ with the property that $\|y^n - z\| \longrightarrow 0$, as a consequence of (1). Since ℓ is continuous, this implies that $\ell(z) = \lim_{n\to\infty} \ell(y^n) = 0$.

Conversely, suppose that $z \notin S$, such that

$$\delta := \inf_{y \in S} \|z - y\| > 0. \tag{2}$$

Consider the subspace Z of all points of the form $y + \alpha z$, with $y \in S$ and $\alpha \in \mathbb{R}$ (assuming for simplicity that X is a real vector space). Define a linear functional $\ell_0 \colon Z \longrightarrow \mathbb{R}$ by

$$\ell_0(y + \alpha z) := \alpha.$$

It follows from (2) that

$$\|y + \alpha z\| = |\alpha|\, \|z - (-y/\alpha)\| \geqslant |\alpha|\delta,$$

so that $|\ell_0(y + \alpha z)| \leqslant \delta^{-1}\|y + \alpha z\|$ for all $y \in S$ and $\alpha \in \mathbb{R}$. By the Hahn-Banach theorem, then ℓ_0 can be extended to a functional on all of X, which we denote by ℓ. By construction, we have $\ell(y_j) = 0$ for all y_j, but $\ell(z) = \delta^{-1} \neq 0$.

Problem 2. A subspace Y in a normed real vector space X is called finite-dimensional if there exists a number $n \in \mathbb{N}$ (the dimension of Y) and vectors $\{y_1, \ldots, y_n\}$ in Y with the property that every element $x \in Y$ has a unique representation of the form

$$x = \alpha_1 y_1 + \ldots + \alpha_n y_n \quad \text{for suitable } \alpha_i \in \mathbb{R} \text{ with } i = 1, \ldots, n. \tag{3}$$

Prove that every finite-dimensional subspace of a normed real vector space is closed.
Hint: Reduce the problem to the fact that $\mathbb{R}^n$ is complete.

Proof. If $Y = \{0\}$, then there is nothing to prove.
Assume therefore that $n \geqslant 1$. Consider the map $c \colon Y \longrightarrow \mathbb{R}^n$ defined by

$$c(\alpha_1 y_1 + \ldots + \alpha_n y_n) := (\alpha_1, \ldots, \alpha_n).$$

This map is an isomorphism because the representation (3) is unique.
Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \Big\| \sum_{i=1}^n \alpha_i y_i \Big\| \leqslant \sum_{i=1}^n |\alpha_i| \leqslant C_2 \Big\| \sum_{i=1}^n \alpha_i y_i \Big\|. \tag{4}$$

In fact, on the one hand we can use the triangle inequality to estimate

$$\Big\| \sum_{i=1}^n \alpha_i y_i \Big\| \leqslant \sum_{i=1}^n |\alpha_i| \|y_i\|,$$

so $C_1 := (\max_i \|y_i\|)^{-1}$ will work for the first inequality in (4). Note that we have $y_i \neq 0$ for all i since otherwise the representation (3) would not be unique.
On the other hand, the map $(\alpha_1, \ldots, \alpha_n) \mapsto \|\alpha_1 y_1 + \ldots + \alpha_n y_n\|$ is continuous because

$$\Big\| \Big(\sum_{i=1}^n \alpha_i y_i \Big) - \Big(\sum_{i=1}^n \beta_i y_i \Big) \Big\| = \Big\| \sum_{i=1}^n (\alpha_1 - \beta_i) y_i \Big\| \leqslant C_1^{-1} \sum_{i=1}^n |\alpha_i - \beta_i|.$$

Therefore we obtain

$$\delta := \inf \left\{ \Big\| \sum_{i=1}^n \alpha_i y_i \Big\| \colon \sum_{i=1}^n |\alpha_i| = 1 \right\} > 0.$$

In fact, the inf of a continuous function over a compact set is attained, and the inf cannot be zero because that would contradict that $y_i \neq 0$ for all i. This implies (4) with $C_2 := \delta^{-1}$.
We have therefore shown that a sequence $\{x^k\}$ in Y converges in the norm $\|\cdot\|$ if and only if the sequence of coefficients $\{c(x^k)\}$ converges in the ℓ^1-norm in $\mathbb{R}^n$. Therefore closedness of Y is equivalent to completeness of $\mathbb{R}^n$ with respect to the ℓ^1-norm. The latter fact is well known.

Problem 3. Let $(X, \mathscr{M}, \mu)$ be a finite measure space.

(1) Prove that the map $d(A, B) := \int_X |\mathbf{1}_A - \mathbf{1}_B| \, d\mu$ defined for all $A, B \in \mathscr{M}$ (with $\mathbf{1}_A$ the characteristic function of A) is a pseudo-distance on the σ-algebra $\mathscr{M}$. That is, the map d satisfies all properties of a distance, except that $d(A, B) = 0$ does not imply $A = B$.

(2) Prove that the pseudo-metric space $(\mathscr{M}, d)$ is complete.

Proof.

(1) We have that

$$0 \leqslant d(A, B) \leqslant \int_X \Big(|\mathbf{1}_A| + |\mathbf{1}_B| \Big) \, d\mu \leqslant \mu(A) + \mu(B) < \infty$$

for all $A, B \in \mathscr{M}$.
The symmetry $d(A, B) = d(B, A)$ is clear from the definition.
To prove the triangle inequality, note that

$$d(A, B) = \mu(A \setminus B) + \mu(B \setminus A).$$

One can then check that

$$\begin{aligned} A \setminus C = A \cap C^c &= \Big(A \cap B^c \cap C^c \Big) \cup \Big(A \cap B \cap C^c \Big) = \Big(A \setminus (B \cup C) \Big) \cup \Big((A \cap B) \setminus C \Big), \\ C \setminus A = C \cap A^c &= \Big(C \cap A^c \cap B^c \Big) \cup \Big(B \cap C \cap A^c \Big) = \Big(C \setminus (A \cup B) \Big) \cup \Big((B \cap C) \setminus A \Big), \end{aligned}$$

and the union is disjoint. Similar formulas hold for A and B interchanged. We find that

$$\begin{aligned} \mu(A \setminus C) + \mu(C \setminus B) &= \mu\Big(A \setminus (B \cup C) \Big) + \mu\Big((A \cap B) \setminus C \Big) + \mu\Big(C \setminus (A \cup B) \Big) + \mu\Big((A \cap C) \setminus B \Big) \\ &= \mu(A \setminus B) + \mu\Big((A \cap B) \setminus C \Big) + \mu\Big(C \setminus (A \cup B) \Big) \end{aligned}$$

because

$$\Big(A \setminus (B \cup C) \Big) \cup \Big((A \cap C) \setminus B \Big) = \Big(A \cap B^c \cap C^c \Big) \cup \Big(A \cap C \cap B^c \Big) = A \cap B^c = A \setminus B,$$

and the union is disjoint. A similar formula holds with A and B interchanged. We conclude that

$$\begin{aligned} d(A, C) + d(B, C) &= \Big(\mu(A \setminus C) + \mu(C \setminus A) \Big) + \Big(\mu(B \setminus C) + \mu(C \setminus B) \Big) \\ &= \mu(A \setminus B) + \mu(B \setminus A) + 2 \Big(\mu\Big((A \cap B) \setminus C \Big) + \mu\Big(C \setminus (A \cup B) \Big) \Big) \\ &\geqslant d(A, B). \end{aligned}$$

(2) Consider a sequence of sets $\{A^n\}$ in $\mathscr{M}$ that is Cauchy with respect to d. Then the sequence $\{\mathbf{1}_{A^n}\}$ is Cauchy with respect to the $\mathscr{L}^1(X, \mu)$-norm, and thus $\mathbf{1}_{A^n} \longrightarrow f$ for some $f \in \mathscr{L}^1(X, \mu)$. Recall that strong convergence implies convergence in measure. We define a map $\varphi(s) := s^2(1-s)^2$ for all $s \in \mathbb{R}$, which vanishes exactly for $s = 0$ or $s = 1$. Then we have that

$$\varphi(\mathbf{1}_{A^n}) \longrightarrow \varphi(f) \quad \text{in measure}.$$

By dominated convergence (note that $0 \leqslant \mathbf{1}_{A^n}(x) \leqslant 1$ for all $x \in X$ and $n \in \mathbb{N}$), we get

$$\int_X \varphi(f) \, d\mu = \lim_{n \to \infty} \int_X \varphi(\mathbf{1}_{A^n}) \, d\mu = 0,$$

and thus $f(x) \in \{0, 1\}$ for μ-a.e. $x \in X$. This implies that $f = \mathbf{1}_A$ for some $A \in \mathscr{M}$. Then

$$d(A^n, A) = \int_X |\mathbf{1}_{A^n} - \mathbf{1}_A| \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty,$$

and so $(\mathscr{A}, d)$ is complete.

Problem 4. Let $(X, \mathscr{M}, \mu)$ be a measure space and consider a sequence of measurable functions $f^n\colon X \longrightarrow \mathbb{R}$. Assume that there exists a function $g \in \mathscr{L}^1(X,\mu)$ with the property that $|f^n| \leqslant g$ for all $n \in \mathbb{N}$. Prove that then

$$\int_X \liminf_{n\to\infty} f^n \, d\mu \leqslant \liminf_{n\to\infty} \int_X f^n \, d\mu \leqslant \limsup_{n\to\infty} \int_X f^n \, d\mu \leqslant \int_X \limsup_{n\to\infty} f^n \, d\mu. \tag{5}$$

Give an example showing that this chain of inequalities may be no longer true if there exists no dominating function g as above.

Proof. Consider the sequence of functions $h^n := f^n - g$. By assumption, we have that $h^n \geqslant 0$ for all $n \in \mathbb{N}$, and so by linearity of the integral and Fatou's lemma we obtain

$$\begin{aligned} \int_X \liminf_{n\to\infty} f^n \, d\mu - \int_X g \, d\mu &= \int_X \liminf_{n\to\infty} h^n \, d\mu \\ &\leqslant \liminf_{n\to\infty} \int_X h^n \, d\mu = \left(\liminf_{n\to\infty} \int_X f^n \, d\mu \right) - \int_X g \, d\mu. \end{aligned}$$

Since $g \in \mathscr{L}^1(X,\mu)$, the last integral is finite, so we can add it on both sides to get

$$\int_X \liminf_{n\to\infty} f^n \, d\mu \leqslant \liminf_{n\to\infty} \int_X f^n \, d\mu.$$

The $\liminf$ is bounded above by the $\limsup$, which proves the second inequality in (5).
To prove the last inequality, we proceed in a similar fashion. Consider the functions $k^n := -f^n + g$. By assumption, we have $k^n \geqslant 0$ for all $n \in \mathbb{N}$, and so by Fatou's lemma we obtain

$$\begin{aligned} \int_X \liminf_{n\to\infty} (-f^n) \, d\mu + \int_X g \, d\mu &= \int_X \liminf_{n\to\infty} k^n \, d\mu \\ &\leqslant \liminf_{n\to\infty} \int_X k^n \, d\mu = \left(\liminf_{n\to\infty} - \int_X f^n \, d\mu \right) + \int_X g \, d\mu. \end{aligned}$$

But now $\liminf\limits_{n\to\infty}(-a^n) = -\limsup\limits_{n\to\infty} a^n$ for any sequence $\{a^n\}$. Therefore

$$-\int_X \limsup_{n\to\infty} f^n \, d\mu \leqslant -\limsup_{n\to\infty} \int_X f^n \, d\mu.$$

which proves the result. We used again that the integral over g is finite.
To prove that the dominated integrability is needed, consider the case $X = \mathbb{R}$, with μ equal to the Lebesgue measure. Consider the functions $f^n := n\mathbf{1}_{(0,1/n)}$ for all $n \in \mathbb{N}$. Then we have

$$\liminf_{n\to\infty} f^n = \limsup_{n\to\infty} f^n = 0 \quad \mu\text{-a.e.},$$

and so

$$\int_X \liminf_{n\to\infty} f^n \, d\mu = \int_X \limsup_{n\to\infty} f^n \, d\mu = 0.$$

On the other hand, we have that

$$\int_X f^n \, d\mu = 1 \quad \text{for all } n \in \mathbb{N}.$$

There exists no dominating function g since such a function would have to behave like $1/x$ as $x \to 0$, which is not integrable.

Problem 5. Let $|\cdot|_e$ denote the exterior (outer) Lebesgue measure on $\mathbb{R}^n$ and let $B(r,x)$ denote the open ball of radius r about $x \in \mathbb{R}^n$. For $E \subset \mathbb{R}^n$ we define outer density $\mathcal{D}_E(x)$ at x by

$$\mathcal{D}_E(x) = \lim_{r\to 0} \frac{|E \cap B(r,x)|_e}{|B(r,x)|_e},$$

whenever the limit exists.

(1) Show that $\mathcal{D}_E(x) = 1$ for a.e. $x \in E$.
(2) Show that E is Lebesgue measurable if and only if $\mathcal{D}_E(x) = 0$ for a.e. $x \in E^c$.

Proof. Notice first that if E is Lebesgue measurable then the function $\chi_E(x)$ is locally integrable and therefore, by the Lebesgue differentiation theorem, $\mathcal{D}_E(x) = \chi_E(x)$ for a.e. $x \in \mathbb{R}^n$. Thus

$$\mathcal{D}_E(x) = \begin{cases} 1 & \text{for a.e. } x \in E \\ 0 & \text{for a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

To prove (1) for arbitrary set E we use the fact that there exists a measurable set U such that $E \subset U$ and for every measurable set M we have

$$|E \cap M|_e = |U \cap M|. \qquad (*)$$

Although the construction of U is more or less standard we sketch it below.

Suppose first $|E|_e < \infty$. For every $n \in \mathbb{N}$ there exists an open set $G_n \supset E$ with $|G_n \setminus E|_e < 1/n$. If $U = \cap_n G_n \supset E$, then $|U| = |E|_e < \infty$ and therefore we can assume that $|M| < \infty$. Note that

$$|E \cap M|_e \geq |M| - |M \setminus (E \cap M)|_e \geq |M| - |M \setminus (U \cap M)| = |U \cap M|,$$

which shows that $(*)$ holds since $|E \cap M|_e \leq |U \cap M|$.

For arbitrary E we can write $E = \cup_k E_k$ where $E_k = E \cap B(0,k)$ has a finite exterior measure. Hence, for every k there exists a measurable set $U_k \supset E_k$ such that $|E_k \cap M|_e = |U_k \cap M|$. Then $U = \liminf U_n = \cup_{n=1}^{\infty} \cap_{k\geq n} U_k \supset E$. If $H_k = \cap_{j\geq k} U_k \subset U_k$ we have $E_k \subset H_k \subset U_k$ which shows that $|E_k \cap M|_e = |H_k \cap M|$. Letting $k \to \infty$ in the last equality we obtain $(*)$.

From $(*)$ and the definition of $\mathcal{D}_E(x)$ it follows that $\mathcal{D}_E(x) = \mathcal{D}_U(x)$ for every $x \in \mathbb{R}^n$ and since $\mathcal{D}_U(x) = 1$ for a.e. $x \in U$ we see that $\mathcal{D}_E(x) = 1$ for a.e. $x \in E$.

To prove (2) it remains to show that if $\mathcal{D}_E(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus E$ then E is measurable. If we assume that E is not measurable and take U as above, then $|U \setminus E|_e > 0$ and we get a contradiction since $\mathcal{D}_E(x) = 1$ for a.e. $x \in U$.

Problem 6.

(1) Let X, Y, Z be Banach spaces and let $B : X \times Y \longrightarrow Z$ be a separately continuous bilinear map, that is, $B(x, \cdot) \in L(Y, Z)$ for each fixed $x \in X$ and $B(\cdot, y) \in L(X, Z)$ for each fixed $y \in Y$. Prove that B is jointly continuous, that is, continuous from $X \times Y$ to Z.

(2) Is there a (nonlinear) function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, which is separately continuous, but not jointly continuous?

Proof.

(1) Denote $B_x = B(x, \cdot) : Y \to Z$ and $B^y = B(\cdot, y) : X \to Z$. Then for every $x \in X$ we have

$$||B^y(x)|| = ||B(x, y)|| = ||B_x(y)|| \leq ||B_x||\,||y||,$$

which shows that

$$\sup_{||y||=1} ||B^y(x)|| \leq ||B_x||.$$

By the uniform boundedness principle we conclude that

$$C = \sup_{||y||=1} ||B^y|| < \infty.$$

For a nonzero $y \in Y$ we put $y' = \frac{1}{||y||} y$ and we see that

$$|B(x, y)|| = ||y||\,||B(x, y')|| = ||y||\,||B^{y'}(x)|| \leq C||x||\,||y||,$$

i.e. $||B(x, y)|| \leq C||x||\,||y||$. Clearly this inequality is also true when $y = 0$. The continuity of B now follows immediately since

$$||B(x, y) - B(x_0, y_0)|| \leq ||B(x, y - y_0)|| + ||B(x - x_0, y_0)|| \leq C\left(||x||\,||y - y_0|| + ||x - x_0||\,||y_0||\right).$$

(2) Yes. The function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous in each variable separately, but is not continuous at the origin.

Problem 7. Let X be a real normed space. Prove that the norm is induced by an inner product if and only if the norm satisfies the parallelogram law, i.e.

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \quad \text{for every } x, y \in X. \qquad (*)$$

Proof. If X is an inner product space then

$$||x+y||^2 = \langle x+y, x+y\rangle = ||x||^2 + ||y||^2 + 2\langle x, y\rangle,$$

and likewise

$$||x-y||^2 = \langle x-y, x-y\rangle = ||x||^2 + ||y||^2 - 2\langle x, y\rangle.$$

Adding these equalities we obtain $(*)$. Conversely, suppose that $(*)$ holds. We want to show that the map $(x,y) \mapsto \langle x, y\rangle$ from $X \times X \longrightarrow \mathbb{R}$ defined by

$$\langle x, y\rangle = \frac{1}{2}(||x+y||^2 - ||x||^2 - ||y||^2) \qquad (6)$$

is an inner product on X. Clearly $\langle x, y\rangle = \langle y, x\rangle$ and $\langle x, x\rangle = ||x||^2 \geq 0$ with equality if and if $x = 0$. Thus, it remains to prove that $\langle x, y\rangle$ is linear in x. Using the definition (6) we see that

$$2\langle x+y, z\rangle - 2\langle x, z\rangle - 2\langle y, z\rangle = (||x+y+z||^2 + ||z||^2) - (||x+z||^2 + ||y+z||^2) + (||x||^2 + ||y||^2 - ||x+y||^2). \qquad (7)$$

On the other hand, from $(*)$ we see that

$$||x+y+z||^2 + ||z||^2 = \frac{||x+y+2z||^2 + ||x+y||^2}{2}$$

$$||x+z||^2 + ||y+z||^2 = \frac{||x+y+2z||^2 + ||x-y||^2}{2}.$$

Substituting these in the right-hand side of (7) we find

$$2\langle x+y, z\rangle - 2\langle x, z\rangle - 2\langle x, y\rangle = ||x||^2 + ||y||^2 - \frac{||x+y||^2}{2} - \frac{||x-y||^2}{2} = 0,$$

proving that

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle x, y\rangle \quad \text{for every } x, y \in X. \qquad (8)$$

It remains to show that

$$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle \quad \text{for every } x, y \in X \text{ and } \alpha \in \mathbb{R}. \qquad (9)$$

From (8) it follows (by induction) that (9) holds when α is a positive integer. Replacing x by $\frac{1}{\alpha}x$ we see that if (9) holds for some $\alpha \neq 0$, then it is true also for $1/\alpha$. Hence (9) holds for all positive rational numbers α. The case $\alpha = -1$ follows easily from $(*)$ thus proving (9) for all $\alpha \in \mathbb{Q}$.

We can prove now (9) by a limiting procedure if we can show that $\langle x, y\rangle$ is a continuous function of x for every y fixed. From the triangle inequality we see that

$$\langle x, y\rangle \leq \frac{1}{2}\left((||x|| + ||y||)^2 - ||x||^2 - ||y||^2\right) = ||x||\,||y||,$$

and therefore

$$|\langle x, y\rangle| = \langle \pm x, y\rangle \leq ||x||\,||y||.$$

For arbitrary $\alpha \in \mathbb{R}$ we consider a sequence of rational numbers $\{r_n\}$ such that $r_n \to \alpha$. Note that $\lim_{n\to\infty}\langle r_n x, y\rangle = \langle \alpha x, y\rangle$, because

$$|\langle r_n x, y\rangle - \langle \alpha x, y\rangle| = |\langle (r_n - \alpha)x, y\rangle| \leq |r_n - \alpha|\,||x||\,||y|| \to 0 \quad \text{as } n \to \infty.$$

The proof of (9) now follows by letting $n \to \infty$ in $\langle r_n x, y\rangle = r_n\langle x, y\rangle$.

Problem 8. Let $(X, \mathscr{M}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence in L^p where $1 < p < \infty$ such that $\sup_n ||f_n||_p < \infty$. Show that if $f_n \to 0$ a.e., then $f_n \to 0$ weakly in L^p.

Proof. Let q be the conjugate exponent to p. Since L^q is the dual of L^p, we must show that $\int f_n g \, d\mu \to 0$ for every $g \in L^q$.

Let $M = \sup_n ||f_n||_p < \infty$. Fix $\epsilon > 0$. Since $d\nu = |g|^q \, d\mu$ is a finite measure on $(X, \mathscr{M})$ that is absolutely continuous with respect to μ, there exists $\delta > 0$ such that

$$\text{if } E \in \mathscr{M} \text{ and } \mu(E) < \delta, \text{ then } \left(\int_E |g|^q \, d\mu \right)^{1/q} < \epsilon.$$

On the other hand, $\mu(X) < \infty$, $f_n \to 0$ a.e. and therefore by Egoroff's theorem there exists $E \in \mathscr{M}$ such that $\mu(E) < \delta$ and $f_n \to 0$ uniformly on $X \setminus E$.

Thus, there is some N such that for $n \geq N$ we have

$$|f_n(x)| < \epsilon \quad \text{for every } x \in X \setminus E.$$

Using the above and Hölder's inequality, we see that for $n \geq N$ we have

$$\begin{aligned} \left| \int f_n g \, d\mu \right| &\leq \int_E |f_n g| \, d\mu + \int_{X \setminus E} |f_n g| \, d\mu \\ &\leq ||f_n||_p \left(\int_E |g|^q \, d\mu \right)^{1/q} + \left(\int_{X \setminus E} |f_n|^p \, d\mu \right)^{1/p} ||g||_q \\ &\leq \left(M + \mu(X)^{1/p} \, ||g||_q \right) \epsilon, \end{aligned}$$

completing the proof.