Algebra Comprehensive Exam

— Fall 2010 —

Instructions: Complete five of the six problems below, and **circle** their numbers exactly in the box below–the uncircled problems will **not** be graded.

1	2	3	4	5	6

(1) (a) Compute the number of p-Sylow subgroups of the alternating group A_5. Justify your answer.

(b) How many elements of order 5, 3, 4 and 2 does A_5 have? Justify your answer.

Solution. (a) $|A_5| = 60 = 3.2.5$, so A_5 has nontrivial p-Sylow subgroups for $p = 2, 3, 5$. Every 5-Sylow subgroup has order 5, and the number of 5-Sylow subgroups is $1 + 5p$ which divides $60/5 = 12$ so it is 1 or 6. Every 5-cycle generates a 5-Sylow subgroup, so there is more than one 5-Sylow, so their number is 6.

Likewise, a 3-Sylow subgroup has order 3, and there are 1,4 or 10 3-Sylow subgroups. By inspection (eg looking at 3-cycles (abc)) the number is more than 4, so it is 10.

For $i \in \{1, 2, 3, 4, 5\}$ let $\{a, b, c, d\}$ denote its complement in $\{1, 2, 3, 4, 5\}$ and consider the subgroup $V_i = \{1, (ab)(cd), (ac)(bd), (ad)(bc)\}$. Every 2-Sylow subgroup has order 4, so each V_i is a 4-Sylow subgroup. It is easy to see that the conjugates of V_1 are V_i for $i = 1, 2, 3, 4, 5$. Since all 2-Sylow subgroups are conjugate, it follows there are exactly 5 2-Sylow subgroups.

(b) Every element of order 5 belongs to a 5-Sylow subgroup. These subgroups are cyclic of order 5, and any two of them intersect trivially. So, there are $6.(5 - 1) = 24$ elements of order 5.

Likewise, there are $10.(3 - 1) = 20$ elements of order 3.

Likewise, there are $5.(4-1) = 15$ elements of order 2. An element of order 4 would belong to a 2-Sylow subgroup, but all those 2-Sylow subgroups have no element of order 4. So, there are no elements of order 4. □

(2) Let X denote the graph which consists of the 1-skeleton of a 3-dimensional cube $[0, 1]^3$. I.e., X contains the vertices and the edges of $[0, 1]^3$. An automorphism f of the graph X is a bijection of the vertices of X that sends edges of X to edges of X. The set of automorphisms of X is a finite group G, under composition.

(a) How many elements does G have? Justify your answer.

(b) Prove that G is not a simple group.

Solution. (a) Consider the action of G on the set of vertices of X. Let a be a vertex of X, and let $\{b, c, d\}$ denote its 3 neighbors. Let H denote the subgroup of G which consists of all automorphisms f that fix a. G acts transitively on the set of vertices of X and the stabilizer of $\{a\}$ is H. Since X has 8 vertices, it follows that $|G| = 8|H|$. Now, every 3-cycle or 2-cycle of $\{b, c, d\}$ can be realized by an element of H thus, H has 6 elements and in fact is isomorphic to S_3. So, $|G| = 48$.

(b) The number of 3-Sylow subgroups of G is $1 + 3k$ and divides 16, so it is 1 or 4. If there are 4 3-Sylow subgroups, then the action of G on the set of 3-Sylow subgroups gives a nontrivial homomorphism $G \to S_4$ so it has nontrivial kernel K. □

(3) Let G be a finite group of n elements and let r be the nuber of conjugacy classes of G. Show that the cardinality of the set

$$X = \{(a, b) \in G \times G | ab = ba\}$$

is nr.

Solution. For fixed $a \in G$, the number of b such that $(a, b) \in X$ is $|C(a)|$ where $C(a)$ are all elements that commute with a. Now sum over a. We get $|X| = \sum_{a \in G} |C(a)|$. Now break the above sum over conjugacy classes, observing that if a and a' are conjugate (ie $a = g^{-1}a'g$), then $C(a) = g^{-1}C(a')g$. If $N(a)$ is the conjugacy class of a, we have $X = \sum |C(a)||N(a)|$ where we sum over conjugacy classes. Now $|C(a)||N(a)| = |G| = n$. The result follows. □

(4) If $\omega = e^{2\pi i/3}$, prove that the ring $R = \mathbb{Z}[\omega]$ is a Euclidean domain, by using the norm $d(x) = x\bar{x}$ for $x \in \mathbb{C}$.

Solution. We need to show that for every $x, y \in R$ with $x \neq 0$ there exist $t, r \in R$ such that $y = tx + r$ and $r = 0$ or $d(r) < d(x)$. First, assume that x is a positive natural number n. Then $y = a + \omega b$ for integers a, b. Set $a = un + u_1$ and $b = vn + v_1$ where u_1, v_1 satisfy $|u_1| \leq n/2$ and $|v_1| \leq n/2$. Then, compute $d(r)$ and confirm OK.

Now, assume x is arbitrary. Then, divide $y\bar{x}$ by $x\bar{x}$ as above ie $y\bar{x} = tx\bar{x} + r$ and write $y = tx + r_0$ where $r_0 = y - tx$. Then, $d(r_0) < d(x)$ or $r_0 = 0$. □

(5) Give an example of a commutative ring with unit element which is a unique factorization domain but not a principal ideal domain.

Solution. Let $R = \mathbb{C}[x, y]$. By a theorem that states S is a unique factorization domain implies $S[x]$ is, it follows R is a unique factorization domain. It is not a principal ideal domain because the ideal (x, y) is not principal (show that). □

(6) Let A be a symmetric $n \times n$ matrix such that $A^2 = J + pI$, where J is the $n \times n$ matrix with all entries equal to 1, I is the $n \times n$ identity matrix, and $p \geq 0$ is a real number. What are the possible eigenvalues of A?

Solution. Let $\vec{x}$ be an eigenvector of A^2 corresponding to the eigenvalue λ. Then

$$\lambda\vec{x} = A^2\vec{x} = (J + pI)\vec{x} = J\vec{x} + p\vec{x},$$

so $J\vec{x} = (\lambda - p)\vec{x}$. Thus, λ is an eigenvalue of A^2 iff $\lambda - p$ is an eigenvalue of J.

Clearly, n is an eigenvalue of J (with eigenvector $(1, \ldots, 1)$). Now J is a symmetric matrix with rank 1; so all other eigenvalues of J are 0. Thus the eigenvalues of A^2 are $n + p$ (with multiplicity 1) and p (with multiplicity $n - 1$). Hence, the possible eigenvalues of A are $\pm\sqrt{n + p}$ and $\pm\sqrt{p}$.

□