1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $\{f_k \colon k \geqslant 1\}$ a sequence of square-integrable functions with the following property: For all $\varepsilon > 0$ there exists an $M_0 \in \mathbb{N}$ so that

$$\Big\| \sup_{M > M_0} \Big| \sum_{k=M_0}^{M} f_k \Big| \Big\|_{\mathscr{L}^2(X,\mu)} < \varepsilon.$$

Show that the series $\sum_{k=1}^{\infty} f_k$ converges a.e.

Solution:

Let $F_n := \sum_{k=1}^{n} f_k$. It is standard to show that the functions $F^* := \limsup_n F_n$ and $F_* := \liminf_n F_n$ are measurable. The claim to be shown is that the set $\{F^* > F_*\}$ has μ-measure zero. Note that this set is equal to the union over $t \in \mathbb{N}$ of the

$$E_t := \Big\{ x \in X \colon \limsup_n F_n > 2^{-t} + \liminf_n F_n \Big\}.$$

Given $\varepsilon > 0$, let M_0 be as in the hypothesis. For any $x \in E_t$, we can choose numbers $n_1, n_2 > M_0$ so that

$$|F_{n_1}(x) - F_{n_2}(x)| = \Big| \sum_{k=1+\min\{n_1,n_2\}}^{\max\{n_1,n_2\}} f_k(x) \Big| > 2^{-t}.$$

This implies that $\sup_{M > M_0} \Big| \sum_{k=M_0}^{M} f_k(x) \Big| > 2^{-t}$.

By Chebyshev inequality, we can then estimate

$$\begin{aligned} \mu(E_t) &\leqslant \mu\Big(\sup_{M>M_0} \Big| \sum_{k=M_0}^{M} f_k \Big| > 2^{-t} \Big) \\ &\leqslant 2^{2t} \Big\| \sup_{M>M_0} \Big| \sum_{k=M_0}^{M} f_k \Big| \Big\|_{\mathscr{L}^2(X,\mu)}^2 \\ &\leqslant \varepsilon^2 2^{2t}. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\mu(E_t) = 0$, hence $\mu\Big(\bigcup_{t=1}^{\infty} E_t\Big) = 0$, by countable subadditivity of μ.

2. Let ν be a signed measure on $I := [0, 1]$ with $|\nu|(I) = 1$ and $\nu(I) = 0$. Suppose that there is a continuous function $f \colon I \longrightarrow [-1, 1]$ so that $\int f \, d\nu = 1$. Show that Lebesgue measure is not continuous with respect to $|\nu|$.

Solution:

We show that there is a non-empty open set U so that $|\nu|(U) = 0$, which certainly is more than enough for the conclusion above.

Appeal to the Jordan decomposition to write $\nu = \nu_+ + \nu_-$, and the Hahn decomposition to write $I = P \cup N$, where

$$\nu_+(P) + \nu_-(N) = 1 \quad \text{and} \quad \nu_+(P) - \nu_-(N) = 0$$

so that $\nu_+(P) = \nu_-(N) = \frac{1}{2}$. Turning to f, since $|f| \leqslant 1$ we have

$$\int f\, d\nu_+ \leqslant \frac{1}{2} \quad \text{and} \quad -\int f\, d\nu_+ \leqslant \frac{1}{2}.$$

But the sum of the two integrals is one, so we must have equality above, and moreover $f = \pm 1$ a.e. $(\nu_\pm)$.

The two measures $\nu_\pm$ are not zero, so f must take the values ± 1. In addition, f is continuous, so $U = f^{-1}(-\frac{1}{2}, \frac{1}{2})$ is open and non-empty. Moreover, we must have

$$|\nu|(U) = \nu_+(U) + \nu_-(U) = 0.$$

3. Let $f\colon [0,1] \longrightarrow [0,1]$ be a Lipschitz function, so that $|f(x) - f(y)| \leqslant C|x - y|$ for some fixed constant C and all $0 \leqslant x, y \leqslant 1$. Let $A \subset [0,1]$ be a Lebesgue measurable set.
   (a) Show that $|f(A)| \leqslant C|A|$, where $|\cdot|$ denotes the Lebesgue measure.
   (b) Show that even if C is optimal, namely $C = \sup_{0 \leqslant x < y \leqslant 1} \frac{|f(x)-f(y)|}{y-x}$, we need not have equality in the first part.

Solution:

(a) Given $\varepsilon > 0$ select a relatively open set $G \subset [0,1]$ with $|G \setminus A| < \varepsilon$. Write the components of G by $G_1, \ldots$. The sets $f(G_k)$ are connected subsets of $[0,1]$, hence they are clopen intervals, and moreover $|f(G_k)| \leqslant C|G_k|$. Thus, we have

$$\begin{aligned} |f(A)| \leqslant |f(G)| &\leqslant \sum_{k=1}^{\infty} |f(G_k)| \\ &\leqslant C \sum_{k=1}^{\infty} |G_k| \\ &\leqslant C(|A| + \varepsilon). \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that $|f(A)| \leqslant C|A|$.

(b) Take

$$f(x) := \begin{cases} x & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < x \leqslant 1 \end{cases}$$

Then, f is Lipschitz with constant one, and $|f([\frac{1}{2}, 1])| = 0$.

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let f_n, $n \geqslant 1$, be a sequence of measurable functions on X so that $f_n \longrightarrow 0$ a.e. and $\sup_n \|f\|_p < \infty$, where $1 \leqslant p < \infty$. Show that for all $g \in \mathscr{L}^q$ with $q = \frac{p}{p-1}$ we have

$$\lim_{n\to\infty} \int f_n \cdot g \, d\mu = 0$$

That is, the functions f_n converges to zero weakly in $\mathscr{L}^p$.

Solution:

We can assume that $f_n \geqslant 0$ and let $g \geqslant 0$. The measure $|g|^q d\nu$ is absolutely continuous with respect to ν. In particular, given $\varepsilon > 0$ we can choose $\delta > 0$ so that

$$\nu(F) < \delta \quad \text{implies} \quad \int_F |g|^q \, d\nu < \varepsilon^q.$$

Now choose n_0 so large that for the event $E = \left\{ \sup_{n \geq n_0} f_n > \varepsilon \right\}$, we have $\mu(E) < \delta$. (This is possible as $\mu(X) < \infty$!)

Then, we estimate using Hölder's inequality,

$$\int_{X \setminus E} f_n g \, d\mu \leqslant \varepsilon \int_{X \setminus E} g \, d\mu \leqslant \varepsilon \mu(X)^{1/p} \|g\|_q.$$

And on the other hand, we can estimate

$$\int_E f_n g \, d\mu \leqslant \sup_n \|f_n\|_p \cdot \left(\int_E |g|^q \, d\mu \right)^{1/q} \leqslant \varepsilon \sup_n \|f_n\|_p.$$

These two inequalities prove the claim.

5. Let X be a normed linear space and X' its dual space. Consider the following statements:

   (a) If X is separable, then X' is separable.

   (b) If X' is separable, then X is separable.

Which statement is true, which one is false? Prove the true statement. Give a counter-example disproving the false statement. Explain why your example works.

Solution:

Statement (b) is true.

To show that (a) is wrong, we consider the case $X = \ell^1$ for which $X' = \ell^\infty$. Assume that ℓ^∞ is separable. That is, assume that there exist countable many sequences

$$a^i = \{a_n^i\}_{n=1}^\infty \in \ell^\infty \quad \text{for } i \in \mathbb{N}$$

that form a dense subset in ℓ^∞. Then we contruct a new sequence $b = \{b_n\}_{n=1}^\infty$ with

$$b_n := a_n^n + 1 \quad \text{for all } n.$$

Then $b \in \ell^\infty$ but $\|b - a^i\| = 1$ for all $i \in \mathbb{N}$, so the a_i are not dense.

To prove (b), assume $\{f_n\}_{n=1}^\infty \subset X'$ is dense in X'. Then the sequence $\{g_n\}_{n=1}^\infty$ with $g_n := f_n/\|f_n\|_{X'}$ (with $f_n \neq 0$) is dense in the unit sphere in X'. Note that

$$\|g_n\|_{X'} = \sup\Big\{|g_n(x)| \colon \|x\|_X = 1\Big\} = 1.$$

Therefore, for any $n \in \mathbb{N}$ there exists an $x_n \in X$ with $\|x_n\|_X = 1$ and $|g_n(x_n)| \geqslant \frac{1}{2}$. Let now S denote the closure of the span of the $\{x_n\}_{n=1}^\infty$, which is separable (consider linear combinations with rational coefficients). Suppose that $S \neq X$. Then we can find a functional $g \in X'$ with $\|g\|_{X'} = 1$ and $g(x) = 0$ for all $x \in S$, by Hahn-Banach theorem. In particular, we would have $g(x_n) = 0$ for all $n \in \mathbb{N}$. But then

$$\frac{1}{2} \leqslant |g_n(x_n)| = |g_n(x) - g(x_n)| \leqslant \|g_n - g\|_{X'}\|x_n\|_X,$$

which implies that $\|g_n - g\|_{X'} \geqslant \frac{1}{2}$ since $\|x_n\|_X = 1$. This is a contradicition to the assumption that the family $\{g_n\}_{n=1}^\infty$ is dense in the unit sphere in X'.

6. Let X be a real Banach space. Consider a countable family $\{x_n\}_{n=1}^\infty$ of elements in X with the following properties:

   (a) The linear span of $\{x_n\}$ is dense in X with respect to the X-norm $\|\cdot\|_X$;

   (b) For any square-summable sequence $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ we have

$$\Big\|\sum_{n=1}^\infty a_n x_n\Big\|_X = \sqrt{\sum_{n=1}^\infty a_n^2}.$$

Prove that the norm $\|\cdot\|_X$ is induced by a scalar product, and thus X is a Hilbert space. Show that $\{x_n\}_{n=1}^\infty$ must then be an orthonormal sequence.

Solution:

We denote by S the linear span of $\{x_n\}_{n=1}^\infty$ (the set of finite linear combinations of elements in $\{x_n\}_{n=1}^\infty$). By property (b), we find that on S the norm $\|\cdot\|_X$ coincides with the ℓ^2-norm of its coefficients. Therefore the closure of S, which is X by assumption (a), is isometrically isomorphic to the closure in ℓ^2 of the set of sequences with only finitely many nonzero terms. The latter closure is ℓ^2. Hence each elements $x \in X$ can be written in the form

$$x = \sum_{n=1}^\infty a_n x_n \quad \text{with } \{a_n\}_{n=1}^\infty \in \ell^2,$$

and $\|x\|_X = \|\{a_n\}_{n=1}^\infty\|_{\ell^2}$. Since ℓ^2 is a Hilbert space, its norm is induced by an inner product and satisfies the parallelogram equality, and so

$$\|x+y\|_X^2 + \|x-y\|_X^2 = 2\Big(\|x\|_X^2 + \|y\|_X^2\Big) \quad \text{for all } x, y \in X.$$

We can then define the inner product using the polarization formula

$$\langle x, y\rangle_X := \frac{1}{4}\Big(\|x+y\|_X^2 - \|x-y\|_X^2\Big) \quad \text{for all } x, y \in X. \tag{1}$$

Note that we are working over the real numbers. One can then check that (1) is an inner product and that $\|x\|_X^2 = \langle x, x\rangle_X$ for all $x \in X$.

For any $i \in \mathbb{N}$ we denote by e^i the sequence whose coefficients vanish everywhere except for the ith entry, which is equal to one. Then we obtain that

$$\begin{aligned}
\langle x_i, x_i\rangle &= \frac{1}{4}\Big(\|x_i + x_i\|_X^2 - \|x_i - x_i\|_X^2\Big) \\
&= \frac{1}{4}\Big(\|e_i + e_i\|_{\ell^2}^2 - \|e_i - e_i\|_{\ell^2}^2\Big) \\
&= \frac{1}{4}\Big(2^2 - 0^2\Big) = 1
\end{aligned}$$

for all $i \in \mathbb{N}$. Similarly, for all $i, j \in \mathbb{N}$ with $i \neq j$ we have

$$\begin{aligned}
\langle x_i, x_j\rangle &= \frac{1}{4}\Big(\|x_i + x_j\|_X^2 - \|x_i - x_j\|_X^2\Big) \\
&= \frac{1}{4}\Big(\|e_i + e_j\|_{\ell^2}^2 - \|e_i - e_j\|_{\ell^2}^2\Big) \\
&= \frac{1}{4}\Big((1^2 + 1^2) - (1^2 + (-1)^2)\Big) = 0.
\end{aligned}$$

This proves that the $\{x_n\}_{n=1}^\infty$ form in fact an orhonormal system.

7. Let (Ω, μ) be a measure space. For some $p \in [1, \infty)$ consider functions $f_n, f \in \mathscr{L}^p(\Omega, \mu)$ with the property that $\|f_n - f\|_{\mathscr{L}^p} \longrightarrow 0$ as $n \to \infty$.

   (a) Prove that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges pointwise a.e. to f.

   (b) Show by example that it is possible that not the whole sequence converges a.e.

Solution:

(a) Convergence in the norm implies convergence in measure: for all $\varepsilon > 0$ we have

$$\mu\Big(\{x \in \Omega \colon |f(x) - f_n(x)| \geqslant \varepsilon\}\Big) \leqslant \frac{1}{\varepsilon^p} \int_\Omega |f - f_n|^p \, d\mu \longrightarrow 0$$

as $n \to \infty$, by Chebyshev's inequality. For any $k \in \mathbb{N}$ we can then find $n_k \in \mathbb{N}$ with

$$\text{for all } n \geqslant n_k \text{ we have } \mu(E_k) \leqslant 2^{-k},$$

where $E_k := \{x \in \Omega \colon |f(x) - f_n(x)| > 1/k\}$. Define $H_m := \bigcup_{k \geqslant m} E_k$ so that

$$\mu(H_m) \leqslant \sum_{k=m}^{\infty} 2^{-k} = 2^{1-m}$$

for all $m \in \mathbb{N}$. If $Z := \bigcap_{m=1}^{\infty} H_m$, then we obtain $\mu(Z) = 0$.

We claim that the $|f(x) - f_{n_k}(x)| \longrightarrow 0$ for all $x \in \Omega \setminus Z$, which will prove the claim. Indeed if $x \notin Z$, then $x \notin H_m$ for some m. Hence $x \notin E_k$ for all $k \geqslant m$ and

$$|f(x) - f_{n_k}(x)| \leqslant 1/k \quad \text{for all } k \geqslant m.$$

This implies precisely that $f_{n_k}(x) \longrightarrow f(x)$ for all $x \notin Z$.

(b) To prove that in general it is necessary to extract a subsequence, let $p = 1$ and $\Omega := [0, 1]$, equipped with the Lebesgue measure. Then we define functions

$$f_n(x) := \mathbf{1}_{[k2^{-N}, (k+1)2^{-N}]}(x) \quad \text{whenever } n = 2^N + k \text{ and } k = 0, \ldots, 2^N - 1$$

for all $n \in \mathbb{N}$. Then $\|f_n\|_{\mathscr{L}^1} \longrightarrow 0$ as $n \to \infty$ and so $f = 0$. Then the subsequence $\{f_{2^k}\}_{k=1}^{\infty}$ converges to zero a.e., whereas the whole sequence does not.

8. Let μ be a regular Borel measure on $\mathbb{R}^n$ and let $V \subset \mathbb{R}^n$ be open. Define $f(x) := \mu(x + V)$ for all $x \in \mathbb{R}^n$.

   (a) Give an example that shows that the function f need not be continuous.

   (b) Prove that f is lower semicontinuous. That is, for all $\alpha > 0$

$$\text{the set } \{x \in \mathbb{R}^n \colon f(x) > \alpha\} \text{ is open.}$$

(c) Prove that if μ is the Lebesgue measure on the open unit ball, then f is continuous.

Solution:

(a) Consider $\mu := \delta_0$ (Dirac measure at the origin) and $V := B_1(0)$ (open unit ball). Then $f(x) = 1$ for all $x \in \mathbb{R}^n$ with $|x| < 1$ and $f(x) = 0$ otherwise.

(b) Fix a point $x \in \mathbb{R}^n$ such that $\gamma := f(x) - \alpha > 0$. Since μ is regular, there exists a compactly supported $g \in \mathscr{C}(\mathbb{R}^n, [0,1])$ such that $\operatorname{spt} g \subset V$ and

$$\mu(x+V) - \gamma/2 \leqslant \int_{\mathbb{R}^n} g(z-x)\,\mu(dz),$$

which implies that

$$\alpha + \gamma/2 \leqslant \int_{\mathbb{R}^n} g(z-x)\,d\mu(dz).$$

Consider now the family of translates $g_y := g(\cdot - y)$ for $y \in \mathbb{R}^n$. Since g is compactly supported and continuous, it is uniformly continuous, which implies that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in \mathbb{R}^n$ with $|y - x| < \delta$ we have

$$|g_y(z) - g_x(z)| < \varepsilon \quad \text{for all } z \in \mathbb{R}^n.$$

In particular, we have that

$$\|g_y - g_x\| \longrightarrow 0 \quad \text{as } |y-x| \to 0,$$

with $\|\cdot\|$ the sup-norm. In particular, we can find $1 > \delta > 0$ such that

$$\left| \int_{\mathbb{R}^n} g_y\,d\mu - \int_{\mathbb{R}^n} g_x\,d\mu \right| \leqslant \|g_y - g_x\| \mu(K) \leqslant \gamma/2$$

for $y \in \mathbb{R}^n$ with $|y-x| < \delta$. Here K denotes a compact set that contains $B_1(x) + \operatorname{spt} g$. Since μ is a regular measure, we have $\mu(K) < \infty$. For such y we then have

$$f(y) = \mu(y+V) \geqslant \int_{\mathbb{R}^d} g_y\,d\mu > \alpha,$$

which proves the claim,

(c) The Lebesgue measure is translation-invariant, therefore f is constant.