1. Let \((X, \mathcal{A}, \mu)\) be a finite measure space, and \(\{f_k : k \geq 1\}\) a sequence of square-integrable functions with the following property: For all \(\varepsilon > 0\) there exists an \(M_0 \in \mathbb{N}\) so that
\[
\left\| \sup_{M > M_0} \sum_{k=M_0}^{M} f_k \right\|_{L^2(X, \mu)} < \varepsilon.
\]
Show that the series \(\sum_{k=1}^{\infty} f_k\) converges a.e.

**Solution:**

Let \(F_n := \sum_{k=1}^{n} f_k\). It is standard to show that the functions \(F^* := \limsup_n F_n\) and \(F_* := \liminf_n F_n\) are measurable. The claim to be shown is that the set \(\{F^* > F_*\}\) has \(\mu\)-measure zero. Note that this set is equal to the union over \(t \in \mathbb{N}\) of the \(E_t := \{x \in X : \limsup_n F_n > 2^{-t} + \liminf_n F_n\}\).

Given \(\varepsilon > 0\), let \(M_0\) be as in the hypothesis. For any \(x \in E_t\), we can choose numbers \(n_1, n_2 > M_0\) so that
\[
|F_{n_1}(x) - F_{n_2}(x)| = \left| \sum_{k=1+\min\{n_1,n_2\}}^{\max\{n_1,n_2\}} f_k(x) \right| > 2^{-t}.
\]

This implies that \(\sup_{M > M_0} \left| \sum_{k=M_0}^{M} f_k(x) \right| > 2^{-t}\).

By Chebyshev inequality, we can then estimate
\[
\mu(E_t) \leq \mu\left( \sup_{M > M_0} \left| \sum_{k=M_0}^{M} f_k \right| > 2^{-t} \right)
\]
\[
\leq 2^{2t} \left\| \sup_{M > M_0} \left| \sum_{k=M_0}^{M} f_k \right| \right\|_{L^2(X, \mu)}^2
\]
\[
\leq \varepsilon^2 2^{2t}.
\]

As \(\varepsilon > 0\) is arbitrary, we conclude that \(\mu(E_t) = 0\), hence \(\mu\left( \bigcup_{t=1}^{\infty} E_t \right) = 0\), by countable subadditivity of \(\mu\).

2. Let \(\nu\) be a signed measure on \(I := [0, 1]\) with \(|\nu|(I) = 1\) and \(\nu(I) = 0\). Suppose that there is a continuous function \(f : I \rightarrow [-1, 1]\) so that \(\int f \, d\nu = 1\). Show that Lebesgue measure is not continuous with respect to \(|\nu|\).
Solution:
We show that there is a non-empty open set $U$ so that $|\nu|(U) = 0$, which certainly is more than enough for the conclusion above.

Appeal to the Jordan decomposition to write $\nu = \nu_+ + \nu_-$, and the Hahn decomposition to write $I = P \cup N$, where

\[ \nu_+(P) + \nu_-(N) = 1 \quad \text{and} \quad \nu_+(P) - \nu_-(N) = 0 \]

so that $\nu_+(P) = \frac{1}{2}$. Turning to $f$, since $|f| \leq 1$ we have

\[ \int f \, d\nu_+ \leq \frac{1}{2} \quad \text{and} \quad -\int f \, d\nu_+ \leq \frac{1}{2}. \]

But the sum of the two integrals is one, so we must have equality above, and moreover $f = \pm 1$ a.e. ($\nu_\pm$).

The two measures $\nu_\pm$ are not zero, so $f$ must take the values $\pm 1$. In addition, $f$ is continuous, so $U = f^{-1}(-\frac{1}{2}, \frac{1}{2})$ is open and non-empty. Moreover, we must have

\[ |\nu|(U) = \nu_+(U) + \nu_-(U) = 0. \]

3. Let $f : [0, 1] \to [0, 1]$ be a Lipschitz function, so that $|f(x) - f(y)| \leq C|x - y|$ for some fixed constant $C$ and all $0 \leq x, y \leq 1$. Let $A \subset [0, 1]$ be a Lebesgue measurable set.

(a) Show that $|f(A)| \leq C|A|$, where $| \cdot |$ denotes the Lebesgue measure.

(b) Show that even if $C$ is optimal, namely $C = \sup_{0 \leq x, y \leq 1} \frac{|f(x) - f(y)|}{y - x}$, we need not have equality in the first part.

Solution:
(a) Given $\varepsilon > 0$ select a relatively open set $G \subset [0, 1]$ with $|G \setminus A| < \varepsilon$. Write the components of $G$ by $G_1, \ldots$. The sets $f(G_k)$ are connected subsets of $[0, 1]$, hence they are clopen intervals, and moreover $|f(G_k)| \leq C|G_k|$. Thus, we have

\[ |f(A)| \leq |f(G)| \leq \sum_{k=1}^{\infty} |f(G_k)| \]

\[ \leq C \sum_{k=1}^{\infty} |G_k| \]

\[ \leq C(|A| + \varepsilon). \]

As $\varepsilon > 0$ was arbitrary, we conclude that $|f(A)| \leq C|A|$. 

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4. Let \((X, \mathcal{A}, \mu)\) be a finite measure space, and let \(f_n, n \geq 1\), be a sequence of measurable functions on \(X\) so that \(f_n \rightarrow 0\) a.e. and \(\sup_n \|f\|_p < \infty\), where \(1 \leq p < \infty\). Show that for all \(g \in \mathcal{L}^q\) with \(q = \frac{p}{p-1}\) we have

\[
\lim_{n \to \infty} \int f_n \cdot g \, d\mu = 0
\]

That is, the functions \(f_n\) converges to zero weakly in \(\mathcal{L}^p\).

**Solution:**

We can assume that \(f_n \geq 0\) and let \(g \geq 0\). The measure \(|g|^q \, d\nu\) is absolutely continuous with respect to \(\nu\). In particular, given \(\varepsilon > 0\) we can choose \(\delta > 0\) so that

\[
\nu(F) < \delta \quad \text{implies} \quad \int_F |g|^q \, d\nu < \varepsilon^q.
\]

Now choose \(n_0\) so large that for the event \(E = \{\sup_{n \geq n_0} f_n > \varepsilon\}\), we have \(\mu(E) < \delta\).

(This is possible as \(\mu(X) < \infty\!\!\)!) Then, we estimate using Hölder’s inequality,

\[
\int_{X \setminus E} f_n g \, d\mu \leq \varepsilon \int_{X \setminus E} g \, d\mu \leq \varepsilon \mu(X)^{1/p} \|g\|_q.
\]

And on the other hand, we can estimate

\[
\int_E f_n g \, d\mu \leq \sup_n \|f_n\|_p \cdot \left(\int_E |g|^q \, d\mu\right)^{1/q} \leq \varepsilon \sup_n \|f_n\|_p.
\]

These two inequalities prove the claim.

5. Let \(X\) be a normed linear space and \(X'\) its dual space. Consider the following statements:

(a) If \(X\) is separable, then \(X'\) is separable.

(b) If \(X'\) is separable, then \(X\) is separable.
Which statement is true, which one is false? Prove the true statement. Give a counter-example disproving the false statement. Explain why your example works.

**Solution:**

Statement (b) is true.

To show that (a) is wrong, we consider the case $\mathbb{X} = \ell^1$ for which $\mathbb{X}' = \ell^\infty$. Assume that $\ell^\infty$ is separable. That is, assume that there exist countable many sequences $a^i = \{a^i_n\}_{n=1}^\infty \in \ell^\infty$ for $i \in \mathbb{N}$ that form a dense subset in $\ell^\infty$. Then we construct a new sequence $b = \{b_n\}_{n=1}^\infty$ with $b_n := a_n^n + 1$ for all $n$.

Then $b \in \ell^\infty$ but $\|b - a^i\| = 1$ for all $i \in \mathbb{N}$, so the $a_i$ are not dense.

To prove (b), assume $\{f_n\}_{n=1}^\infty \subset \mathbb{X}'$ is dense in $\mathbb{X}'$. Then the sequence $\{g_n\}_{n=1}^\infty$ with $g_n := f_n/\|f_n\|_{X'}$ (with $f_n \neq 0$) is dense in the unit sphere in $\mathbb{X}'$. Note that $\|g_n\|_{X'} = \sup \{ |g_n(x)| : \|x\|_{X} = 1 \} = 1$.

Therefore, for any $n \in \mathbb{N}$ there exists an $x_n \in \mathbb{X}$ with $\|x_n\|_X = 1$ and $|g_n(x_n)| \geq \frac{1}{2}$.

Let now $S$ denote the closure of the span of the $\{x_n\}_{n=1}^\infty$, which is separable (consider linear combinations with rational coefficients). Suppose that $S \neq \mathbb{X}$. Then we can find a functional $g \in \mathbb{X}'$ with $\|g\|_{X'} = 1$ and $g(x) = 0$ for all $x \in S$, by Hahn-Banach theorem. In particular, we would have $g(x_n) = 0$ for all $n \in \mathbb{N}$. But then

$$\frac{1}{2} \leq |g_n(x_n)| = |g_n(x) - g(x_n)| \leq \|g_n - g\|_{X'} \|x_n\|_{X},$$

which implies that $\|g_n - g\|_{X'} \geq \frac{1}{2}$ since $\|x_n\|_{X} = 1$. This is a contradiction to the assumption that the family $\{g_n\}_{n=1}^\infty$ is dense in the unit sphere in $\mathbb{X}'$.

6. Let $\mathbb{X}$ be a real Banach space. Consider a countable family $\{x_n\}_{n=1}^\infty$ of elements in $\mathbb{X}$ with the following properties:

(a) The linear span of $\{x_n\}$ is dense in $\mathbb{X}$ with respect to the $\mathbb{X}$-norm $\| \cdot \|_\mathbb{X}$;

(b) For any square-summable sequence $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ we have

$$\left\| \sum_{n=1}^\infty a_n x_n \right\|_\mathbb{X} = \sqrt{\sum_{n=1}^\infty a_n^2}.$$
Prove that the norm $\| \cdot \|_X$ is induced by a scalar product, and thus $X$ is a Hilbert space. Show that $\{x_n\}_{n=1}^\infty$ must then be an orthonormal sequence.

**Solution:**

We denote by $S$ the linear span of $\{x_n\}_{n=1}^\infty$ (the set of finite linear combinations of elements in $\{x_n\}_{n=1}^\infty$). By property (b), we find that on $S$ the norm $\| \cdot \|_X$ coincides with the $\ell^2$-norm of its coefficients. Therefore the closure of $S$, which is $X$ by assumption (a), is isometrically isomorphic to the closure in $\ell^2$ of the set of sequences with only finitely many nonzero terms. The latter closure is $\ell^2$. Hence each element $x \in X$ can be written in the form

$$x = \sum_{n=1}^\infty a_n x_n \quad \text{with} \quad \{a_n\}_{n=1}^\infty \in \ell^2,$$

and $\|x\|_X = \|\{a_n\}_{n=1}^\infty\|_{\ell^2}$. Since $\ell^2$ is a Hilbert space, its norm is induced by an inner product and satisfies the parallelogram equality, and so

$$\|x + y\|_X^2 + \|x - y\|_X^2 = 2\left(\|x\|_X^2 + \|y\|_X^2\right) \quad \text{for all} \quad x, y \in X.$$

We can then define the inner product using the polarization formula

$$\langle x, y \rangle_X := \frac{1}{4} \left(\|x + y\|_X^2 - \|x - y\|_X^2\right) \quad \text{for all} \quad x, y \in X. \quad (1)$$

Note that we are working over the real numbers. One can then check that (1) is an inner product and that $\|x\|_X^2 = \langle x, x \rangle_X$ for all $x \in X$.

For any $i \in \mathbb{N}$ we denote by $e_i$ the sequence whose coefficients vanish everywhere except for the $i$th entry, which is equal to one. Then we obtain that

$$\langle x_i, x_i \rangle = \frac{1}{4} \left(\|x_i + x_i\|_X^2 - \|x_i - x_i\|_X^2\right) = \frac{1}{4} \left(\|e_i + e_i\|_{\ell^2}^2 - \|e_i - e_i\|_{\ell^2}^2\right) = \frac{1}{4} \left(2^2 - 0^2\right) = 1$$

for all $i \in \mathbb{N}$. Similarly, for all $i, j \in \mathbb{N}$ with $i \neq j$ we have

$$\langle x_i, x_j \rangle = \frac{1}{4} \left(\|x_i + x_j\|_X^2 - \|x_i - x_j\|_X^2\right) = \frac{1}{4} \left(\|e_i + e_j\|_{\ell^2}^2 - \|e_i - e_j\|_{\ell^2}^2\right) = \frac{1}{4} \left((1^2 + 1^2) - (1^2 + (-1)^2)\right) = 0.$$

This proves that the $\{x_n\}_{n=1}^\infty$ form in fact an orthonormal system.
7. Let \((\Omega, \mu)\) be a measure space. For some \(p \in [1, \infty)\) consider functions \(f_n, f \in L^p(\Omega, \mu)\) with the property that \(\|f_n - f\|_{L^p} \to 0\) as \(n \to \infty\).

(a) Prove that there exists a subsequence \(\{f_{n_k}\}_{k=1}^\infty\) that converges pointwise a.e. to \(f\).

(b) Show by example that it is possible that not the whole sequence converges a.e.

\[\text{Solution:}\]

(a) Convergence in the norm implies convergence in measure: for all \(\varepsilon > 0\) we have

\[\mu\left(\{x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon\}\right) \leq \frac{1}{\varepsilon^p} \int_\Omega |f - f_n|^p \, d\mu \to 0\]

as \(n \to \infty\), by Chebyshev’s inequality. For any \(k \in \mathbb{N}\) we can then find \(n_k \in \mathbb{N}\) with

\[\text{for all } n \geq n_k \text{ we have } \mu(E_k) \leq 2^{-k},\]

where \(E_k := \{x \in \Omega : |f(x) - f_n(x)| > 1/k\}\). Define \(H_m := \bigcup_{k \geq m} E_k\) so that

\[\mu(H_m) \leq \sum_{k=m}^\infty 2^{-k} = 2^{1-m}\]

for all \(m \in \mathbb{N}\). If \(Z := \bigcap_{m=1}^\infty H_m\), then we obtain \(\mu(Z) = 0\).

We claim that the \(|f(x) - f_{n_k}(x)| \to 0\) for all \(x \in \Omega \setminus Z\), which will prove the claim. Indeed if \(x \notin Z\), then \(x \notin H_m\) for some \(m\). Hence \(x \notin E_k\) for all \(k \geq m\) and

\[|f(x) - f_{n_k}(x)| \leq 1/k \quad \text{for all } k \geq m.\]

This implies precisely that \(f_{n_k}(x) \to f(x)\) for all \(x \notin Z\).

(b) To prove that in general it is necessary to extract a subsequence, let \(p = 1\) and \(\Omega := [0, 1]\), equipped with the Lebesgue measure. Then we define functions

\[f_n(x) := 1_{[k2^{-N},(k+1)2^{-N}]}(x) \quad \text{whenever } n = 2^N + k \text{ and } k = 0, \ldots, 2^N - 1\]

for all \(n \in \mathbb{N}\). Then \(\|f_n\|_{L^1} \to 0\) as \(n \to \infty\) and so \(f = 0\). Then the subsequence \(\{f_{2^k}\}_{k=1}^\infty\) converges to zero a.e., whereas the whole sequence does not.

8. Let \(\mu\) be a regular Borel measure on \(\mathbb{R}^n\) and let \(V \subset \mathbb{R}^n\) be open. Define \(f(x) := \mu(x + V)\) for all \(x \in \mathbb{R}^n\).

(a) Give an example that shows that the function \(f\) need not be continuous.

(b) Prove that \(f\) is lower semicontinuous. That is, for all \(\alpha > 0\)

the set \(\{x \in \mathbb{R}^n : f(x) > \alpha\}\) is open.
(c) Prove that if $\mu$ is the Lebesgue measure on the open unit ball, then $f$ is continuous.

Solution:

(a) Consider $\mu := \delta_0$ (Dirac measure at the origin) and $V := B_1(0)$ (open unit ball). Then $f(x) = 1$ for all $x \in \mathbb{R}^n$ with $|x| < 1$ and $f(x) = 0$ otherwise.

(b) Fix a point $x \in \mathbb{R}^n$ such that $\gamma := f(x) - \alpha > 0$. Since $\mu$ is regular, there exists a compactly supported $g \in \mathcal{C}(\mathbb{R}^n, [0, 1])$ such that $\text{spt } g \subset V$ and

$$\mu(x + V) - \gamma/2 \leq \int_{\mathbb{R}^n} g(z - x) \mu(dz),$$

which implies that

$$\alpha + \gamma/2 \leq \int_{\mathbb{R}^n} g(z - x) d\mu(dz).$$

Consider now the family of translates $g_y := g(\cdot - y)$ for $y \in \mathbb{R}^n$. Since $g$ is compactly supported and continuous, it is uniformly continuous, which implies that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in \mathbb{R}^n$ with $|y - x| < \delta$ we have

$$|g_y(z) - g_x(z)| < \varepsilon \quad \text{for all } z \in \mathbb{R}^n.$$

In particular, we have that

$$\|g_y - g_x\| \rightarrow 0 \quad \text{as } |y - x| \rightarrow 0,$$

with $\| \cdot \|$ the sup-norm. In particular, we can find $1 > \delta > 0$ such that

$$\left| \int_{\mathbb{R}^n} g_y d\mu - \int_{\mathbb{R}^n} g_x d\mu \right| \leq \|g_y - g_x\| \mu(K) \leq \gamma/2$$

for $y \in \mathbb{R}^n$ with $|y - x| < \delta$. Here $K$ denotes a compact set that contains $B_1(x) + \text{spt } g$. Since $\mu$ is a regular measure, we have $\mu(K) < \infty$. For such $y$ we then have

$$f(y) = \mu(y + V) \geq \int_{\mathbb{R}^d} g_y d\mu > \alpha,$$

which proves the claim,

(c) The Lebesgue measure is translation-invariant, therefore $f$ is constant.