Georgia Institute of Technology
School of Mathematics
Algebra Comprehensive Exam
Fall 2011

Solve any 5 of the following 8 problems. Indicate clearly which 5 problems you would like to be graded.

1. Let A be an $n \times n$ matrix with coefficients in the complex numbers. Prove that if there is an integer $m > 1$ such that $A^m = A$ then A is diagonalizable.
Solution: Let T be a matrix so that $J = TAT^{-1}$ is in Jordan form. We need to prove that J is diagonal. Note that

$$J^m = (TAT^{-1})^m = TA^mT^{-1} = TAT^{-1} = J.$$

The Jordan blocks of J^m are the powers of the blocks of J, so it will suffice to prove that if B is a Jordan block and $B^m = B$, then B is diagonal (and so 1×1). Suppose on the contrary that $n > 1$ and B is an $n \times n$ Jordan block with eigenvalue λ. We will show that $B^m \neq B$ for all $m > 1$. First, an easy induction argument shows that for $m > 1$,

$$B^m = \begin{pmatrix} \lambda^m & m\lambda^{m-1} & \cdots \\ 0 & \lambda^m & \cdots \\ \cdots & & \end{pmatrix}.$$

If $\lambda^m = \lambda$ then either $\lambda = 0$ and the $(1,2)$ entry of B^m is 0, or $\lambda^{m-1} = 1$ and the $(1,2)$ entry is m. In both cases, the $(1,2)$ entry is not 1 and so $B^m \neq B$ for all $m > 1$.

2. Let V be the vector space of polynomials of degree ≤ 3 with complex coefficients. Let $v_i = (x-1)^i$ so that $v_0, \ldots, v_3$ forms a basis of V. Let $f_0, \ldots, f_3$ be the dual basis of the dual vector space V^*. Consider the element F of V^* defined by

$$F(p(x)) = \int_{-1}^{1} p(x)\,dx.$$

Write F in terms of $f_0, \ldots, f_3$.
Solution: Straightforward calculation shows that $F(v_0) = 2$, $F(v_1) = -2$, $F(v_2) = 8/3$, and $F(v_3) = -4$. Thus

$$F = 2f_0 - 2f_1 + (8/3)f_2 - 4f_3.$$

3. Let G be a finite group, and N a normal subgroup of G. Suppose that N has the property that the natural homomorphism $N \to \mathrm{Aut}(N)$ (sending an element to its action by conjugation) is an isomorphism. Show that there exists a subgroup H of G so that $G = N \times H$.
Solution: Let $g_1, \ldots, g_n$ be a set of coset representatives for G/N. For each i, define an automorphism $s(g_i)$ of N by the formula $s(g_i)(n) = g_i^{-1} n g_i$. (Here we use that N is a normal subgroup of G.) By hypothesis, there is a unique element n_i of N such that $s(g_i)(n) = n_i^{-1} n n_i$ for all $n \in N$. Let $t(g_i) = g_i n_i^{-1}$. Then $t(g_i)$ is in the coset $g_i N$ and it is the unique element of this coset which centralizes N. It follows that t defines a homomorphism $G/N \to G$. Letting H be the image of this homomorphism, we have $G \cong N \times H$.

4. Let G be the abelian group with generators x, y, and z and relations

$$2x - 3y + 4z = 2x + 2y + 2z = 6x - 4y + 10z = 0.$$

What is the rank of G? What is the structure of the torsion subgroup of G?

Solution: By hypothesis, G is the cokernel of the homomorphism $\mathbf{Z}^3 \to \mathbf{Z}^3$ given by the matrix

$$\begin{pmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 6 & -4 & 10 \end{pmatrix}.$$

A sequence of elementary integer row and column operations (equivalently, changing basis in the domain and range) yields the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

It follows that $G \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. It has rank 1 and its torsion subgroup is cyclic of order 2.

5. Let R be the subring of $\mathbf{Z}[x]$ consisting of polynomials where the coefficients of x and x^2 are zero.
a) Show that $\mathbf{Q}(x)$ is the field of fractions of R.
b) Compute the integral closure of R in $\mathbf{Q}(x)$. (Recall that the integral closure of R in $\mathbf{Q}(x)$ is defined to be the set of elements of $\mathbf{Q}(x)$ which are roots of a monic polynomial with coefficients in R. You may use the standard fact that the integral closure of R in $\mathbf{Q}(x)$ is a subring of $\mathbf{Q}(x)$.)
Solution: (a) Let F be the field of fractions of R. It is clear that $F \subset \mathbf{Q}(x)$. On the other hand, if $r = f(x)/g(x) \in \mathbf{Q}(x)$ is a ratio of polynomials, then writing

$$r = \frac{x^3 f(x)}{x^3 g(x)}$$

shows that r is a ratio of elements of R. This proves that $R = \mathbf{Q}(x)$.

(b) Let S be the integral closure of R in $\mathbf{Q}(x)$. The polynomial $T^3 - x^3$ is monic with coefficients in R and has x as a root, so $x \in S$. Since S is a ring containing R, we have $\mathbf{Z}[x] \subset S$. It is a standard result that $\mathbf{Z}[x]$ is integrally closed, so we have $S = \mathbf{Z}[x]$.

6. Let R be the ring $(\mathbf{Z}/20\mathbf{Z})[x]$. List the prime and maximal ideals of R.
Solution: Note that by the chinese remainder theorem,

$$R \cong (\mathbf{Z}/4\mathbf{Z})[x] \oplus (\mathbf{Z}/5\mathbf{Z})[x].$$

Note also that $(2, 0) \in R$ is nilpotent, so contained in every prime ideal. Thus the prime ideals of R are in bijection (via inverse image) with the prime ideals of the quotient

$$S = (\mathbf{Z}/2\mathbf{Z})[x] \oplus (\mathbf{Z}/5\mathbf{Z})[x]$$

of R by the ideal generated by $(2, 0)$.

If $M \subset S$ is a maximal ideal, the quotient S/M must be a finite field (since S is finitely generated) necessarily of characteristic 2 or 5. It follows that M has the form $f(\mathbf{Z}/2\mathbf{Z})[x] \oplus (\mathbf{Z}/5\mathbf{Z})[x]$ or $(\mathbf{Z}/2\mathbf{Z})[x] \oplus g(\mathbf{Z}/5\mathbf{Z})[x]$ where f (resp. g) is an irreducible monic polynomial in $(\mathbf{Z}/2\mathbf{Z})[x]$ (resp. $(\mathbf{Z}/5\mathbf{Z})[x]$). The ideals of S which are prime but not maximal are $\{0\} \oplus (\mathbf{Z}/5\mathbf{Z})[x]$ and $(\mathbf{Z}/2\mathbf{Z})[x] \oplus \{0\}$.

7. Let F be an infinite field and let K be a finite extension of F. Assume there are only finitely many subfields L so that $F \subset L \subset K$. Show that K is a primitive extension of F, that is, $K = F(\theta)$, where $\theta \in K$.

Solution: Consider K as a vector space over F. Since F is infinite, K is not the union of any finite collection of proper subspaces. This implies that there is an element $\theta \in K$ which does not lie in any field L with $F \subset L \subset K$ with $L \neq K$. It is then immediate that $K = F(\theta)$.

8. Let p be a prime number and let $\mathbf{F}_p$ be the field of p elements. Find the number of monic irreducible polynomials of degree 6 with coefficients in $\mathbf{F}_p$.
Solution: If f is a monic irreducible in $\mathbf{F}_p[x]$ of degree 6, then each root of f in $\overline{\mathbf{F}}_p$ generates $\mathbf{F}_{p^6}$ over $\mathbf{F}_p$. Conversely, given an element $x \in \mathbf{F}_{p^6}$ which does not lie in any smaller field, the minimal polynomial of x over $\mathbf{F}_p$ is monic irreducible of degree 6. The number of elements in $\mathbf{F}_{p^6}$ not contained in a smaller field is $p^6 - p^3 - p^2 + p$ and so the number of monic irreducible polynomials of degree 6 in $\mathbf{F}_p[x]$ is

$$\frac{p^6 - p^3 - p^2 + p}{6}.$$