Analysis Comprehensive Exam Questions
Fall 2012

Instructions: Complete 5 of the 8 problems below. If you attempt more than five questions, then clearly indicate which five should be graded.

NOTE: Throughout this exam, the Lebesgue exterior measure of a set $E \subseteq \mathbb{R}^d$ will denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is denoted by $|E|$. The characteristic function of a set A is denoted by χ_A.

1. Given a set $E \subseteq \mathbb{R}^d$ with $|E|_e < \infty$, show that E is Lebesgue measurable if and only if for each $\varepsilon > 0$ we can write $E = (S \cup A) \setminus B$ where S is a union of finitely many nonoverlapping boxes and $|A|_e, |B|_e < \varepsilon$.
Remark: A *box* is a rectangular parallelepiped of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$. Boxes are nonoverlapping if their interiors are disjoint.

Solution: $\Rightarrow$. Suppose that E is measurable, and fix $\varepsilon > 0$. Then there exists an open set $U \supseteq E$ such that $|U \setminus E|_e < \varepsilon$. Since U is open, there exist nonoverlapping boxes Q_k such that $U = \cup_{k=1}^{\infty} Q_k$. Since

$$\sum_{k=1}^{\infty} |Q_k| = |U| < \infty,$$

we can choose M large enough that $\sum_{k=M+1}^{\infty} |Q_k| < \varepsilon$. Let

$$S = \bigcup_{k=1}^{M} Q_k, \qquad A = E \setminus S, \qquad B = S \setminus E.$$

Note that S is a finite union of nonoverlapping boxes. Since

$$A = E \setminus S \subseteq U \setminus S \subseteq \bigcup_{k=M+1}^{\infty} Q_k,$$

we have

$$|A|_e \leq |U \setminus S| \leq \Big| \bigcup_{k=M+1}^{\infty} Q_k \Big| \leq \sum_{k=M+1}^{\infty} |Q_k| < \varepsilon.$$

Finally, $B = S \setminus E \subseteq U \setminus E$, so

$$|B|_e \leq |U \setminus E|_e < \varepsilon.$$

$\Leftarrow$. Fix $\varepsilon > 0$. By hypothesis, $E = (S \cup A) \setminus B$, where S is a finite union of nonoverlapping boxes and $|A|_e, |B|_e < \varepsilon$. Since S is measurable, let $U \supseteq S$ be an open set such that $|U\backslash S| < \varepsilon$. Although we don't know that A is measurable, we can find an open set $V \supseteq A$ such that $|V| \leq |A|_e + \varepsilon$. Consequently,

$$|V| \ \leq \ |A|_e + \varepsilon \ < \ 2\varepsilon.$$

Let $G = U \cup V$. Then G is open, and since $U \supseteq S$ and $V \supseteq A$, we have that $G \supseteq S \cup A \supseteq E$. After some set-theoretic calculations, we see that

$$G\backslash E \ \subseteq \ (U\backslash S) \cup V \cup B.$$

Therefore

$$|G\backslash E|_e \ \leq \ |U\backslash S| + |V| + |B|_e \ \leq \ \varepsilon + 2\varepsilon + \varepsilon \ = \ 4\varepsilon,$$

so E is measurable.

2. (a) Let $E \subseteq \mathbb{R}^d$ be a measurable set such that $|E| < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on E, and suppose that f_n is finite a.e. for each n. Show that if $f_n \to f$ a.e. on E, then $f_n \overset{\mathrm{m}}{\to} f$.

(b) Show by example that part (a) can fail if $|E| = \infty$.

Solution: (a) The function f is measurable since it is the pointwise a.e. limit of the measurable functions f_n. Fix $\varepsilon > 0$. We want to show that $|\{|f - f_n| \geq \varepsilon\}| \to 0$ as $n \to \infty$.

Fix $\eta > 0$. By Egorov's Theorem (which is applicable since E has finite measure), there exists a set $A \subseteq E$ such that $|A| < \eta$ and $f_n \to f$ uniformly on $E \backslash A$. Hence there exists an integer $N > 0$ such that

$$\forall\, n > N, \quad \sup_{x \notin A} |f(x) - f_n(x)| \;<\; \varepsilon.$$

Therefore, if $n > N$ and $|f(x) - f_n(x)| \geq \varepsilon$, then $x \in A$. In other words, $\{|f - f_n| \geq \varepsilon\} \subseteq A$ for all $n > N$. Hence for all $n > N$ we have

$$\big|\{|f - f_n| \geq \varepsilon\}\big| \;\leq\; |A| \;<\; \eta.$$

This shows that

$$\lim_{n\to\infty} \big|\{|f - f_n| \geq \varepsilon\}\big| \;=\; 0.$$

(b) Set $f_n = \chi_{[n,n+1]}$. Then $f_n \to 0$ pointwise on $\mathbb{R}$, but f_n does not converge in measure to the zero function. Another example is $f_n(x) = x/n$.

3. Suppose that f is a bounded, real valued, measurable function on $[0,1]$ such that $\int_0^1 x^n f(x)\,dx = 0$ for all $n = 0, 1, 2, \ldots$. Show that $f(x) = 0$ almost everywhere.

Solution: Note the hypothesis $\int_0^1 x^n f(x)dx = 0$ implies that $\int_0^1 p(x)f(x)dx = 0$ for every polynomial p. Fix $\varphi \in C[0,1]$. By the Weierstrass Approximation Theorem, there exist polynomials p_n such that $\|p_n - \varphi\|_\infty \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned} \left|\int_0^1 f(x)\,\varphi(x)\,dx\right| &\leq \left|\int_0^1 f(x)\,(\varphi(x) - p_n(x))\,dx\right| + \left|\int_0^1 f(x)\,p_n(x)\,dx\right| \\ &= \left|\int_0^1 f(x)\,(\varphi(x) - p_n(x))\,dx\right| \\ &\leq \|f\|_1\,\|\varphi - p_n\|_\infty \to 0. \end{aligned}$$

Note that we have $\|f\|_1 \leq \|f\|_\infty$ since we are on a finite measure space. Thus, we have that

$$\int_0^1 f(x)\varphi(x)dx = 0$$

for every continuous function φ. Since $C[0,1]$ is dense in $L^1[0,1]$ we can select a sequence $\{\varphi_n\}$ of continuous functions such that $\|f - \varphi_n\|_1 \to \infty$. Then we have

$$\begin{aligned} \int_0^1 f(x)^2\,dx &\leq \left|\int_0^1 f(x)\,(f(x) - \varphi_n(x))\,dx\right| + \left|\int_0^1 f(x)\,\varphi_n(x)\,dx\right| \\ &= \left|\int_0^1 f(x)\,(f(x) - \varphi_n(x))\,dx\right| \\ &\leq \|f\|_\infty\,\|\varphi - p_n\|_1 \to 0. \end{aligned}$$

Therefore

$$\int_0^1 f(x)^2\,dx = 0,$$

so $f(x) = 0$ almost everywhere as claimed.

4. Fix $1 \le p < \infty$. Given $f_n,\ f \in L^p(\mathbb{R}^d)$, prove that $\|f - f_n\|_p \to 0$ if and only if the following three conditions hold.

(a) $f_n \overset{\mathrm{m}}{\to} f$.

(b) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every measurable set $E \subseteq \mathbb{R}^d$ satisfying $|E| < \delta$ we have $\int_E |f_n|^p < \varepsilon$ for every n.

(c) For each $\varepsilon > 0$ there exists a measurable set $E \subseteq \mathbb{R}^d$ such that $|E| < \infty$ and $\int_{E^{\mathrm{C}}} |f_n|^p < \varepsilon$ for every n.

Solution: $\Rightarrow$. Assume that $\|f - f_n\|_p \to 0$. We must show that conditions (a), (b), and (c) hold.

(a) Tchebyshev's Inequality implies that if a sequence converges in L^p-norm then it converges in measure.

(b) Fix $\varepsilon > 0$. Since $|f|^p$ is integrable, there exists a $\delta_0 > 0$ such that

$$|E| \;<\; \delta_0 \quad\Longrightarrow\quad \|f\,\chi_E\|_p^p \;=\; \int_E |f|^p \;<\; \frac{\varepsilon}{2^{p+1}}.$$

Further, there is some $N > 0$ such that

$$n > N \quad\Longrightarrow\quad \|f - f_n\|_p^p \;<\; \frac{\varepsilon}{2^{p+1}}.$$

Hence if $n > N$ and $|E| < \delta_0$ then we have

$$\begin{aligned}
\int_E |f_n|^p \;&=\; \|(f_n - f + f)\,\chi_E\|_p^p \\
&\le\; 2^p\,\|(f_n - f)\,\chi_E\|_p^p \;+\; 2^p\|f\,\chi_E\|_p^p \\
&\le\; 2^p\,\|f_n - f\|_p^p \;+\; 2^p\,\frac{\varepsilon}{2^{p+1}} \\
&<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon.
\end{aligned}$$

Since $f_1, \dots, f_N$ are all integrable, for each $n = 1, \dots, N$, there is some $\delta_n > 0$ such that

$$|E| \;<\; \delta_n \quad\Longrightarrow\quad \|f_n\,\chi_E\|_p^p \;=\; \int_E |f_n|^p \;<\; \varepsilon.$$

Therefore if we take $\delta = \min\{\delta_0, \delta_1, \dots, \delta_N\}$, then we have shown that statement (b) holds.

(c) Fix $\varepsilon > 0$. Since $|f|^p$ is integrable, by setting $E = B_r(0)$ with r large enough we will have

$$\|f\,\chi_{E^{\mathrm{C}}}\|_p^p \;=\; \int_{E^{\mathrm{C}}} |f|^p \;<\; \frac{\varepsilon}{2^{p+1}}.$$

There is some $N > 0$ such that

$$n > N \implies \|f - f_n\|_p^p < \frac{\varepsilon}{2^{p+1}}.$$

Hence for all $n > N$ we have

$$\begin{aligned}
\|f\,\chi_{E^{\mathrm{C}}}\|_p^p &\le \|(f_n - f + f)\,\chi_{E^{\mathrm{C}}}\|_p^p \\
&\le 2^p\,\|(f_n - f)\,\chi_{E^{\mathrm{C}}}\|_p^p + 2^p\|f\,\chi_{E^{\mathrm{C}}}\|_p^p \\
&\le 2^p\,\|f_n - f\|_p^p + 2^p\,\frac{\varepsilon}{2^{p+1}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Since $f_1, \dots, f_N$ are all integrable, we can take r large enough that we will also have

$$\|f_n\,\chi_{E^{\mathrm{C}}}\|_p^p < \varepsilon, \qquad n = 1, \dots, N.$$

Therefore statement (c) holds.

$\Leftarrow$. Assume that statements (a), (b), and (c) hold. Fix $\varepsilon > 0$, let E be the set given by statement (c), and let $\delta > 0$ be the number given by statement (b). Statement (a) tells us that $f_n \overset{\mathrm{m}}{\to} f$. Setting

$$A_n = \left\{|f - f_n| > \frac{\varepsilon^{1/p}}{|E|^{1/p}}\right\},$$

there must be some $N > 0$ such that

$$n > N \implies |A_n| < \delta.$$

Applying statement (b), we have

$$n > N \implies \int_{A_n} |f - f_n| < \varepsilon.$$

Putting this all together, for $n > N$ we have

$$\begin{aligned}
\|f - f_n\|_p^p &= \int_{E \cap A_n} |f - f_n|^p + \int_{E \setminus A_n} |f - f_n|^p + \int_{E^{\mathrm{C}}} |f - f_n|^p \\
&\le \varepsilon + \int_{E \setminus A_n} \frac{\varepsilon}{|E|} + \varepsilon \\
&\le 3\varepsilon.
\end{aligned}$$

Therefore we have shown that $f_n \to f$ in L^p-norm.

5. Let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2[a,b]$. Prove that $\{f_n\}_{n\in\mathbb{N}}$ is complete in $L^2[a,b]$ if and only if

$$\sum_{n=1}^{\infty} \left| \int_a^x f_n(t)\,dt \right|^2 \;=\; x-a, \qquad x \in [a,b].$$

Remark: A sequence is *complete* if its finite linear span is dense.

Solution: $\Rightarrow$. If $\{f_n\}$ is complete, then we have by Plancherel's Equality that

$$\begin{aligned}\sum_{n=1}^{\infty} \left| \int_a^x f_n(t)\,dt \right|^2 \;&=\; \sum_{n=1}^{\infty} |\langle \chi_{[a,x]}, f_n\rangle|^2 \\ &=\; \|\chi_{[a,x]}\|_2^2 \;=\; \int_a^b |\chi_{[a,x]}(t)|^2\,dt \;=\; x-a.\end{aligned}$$

$\Leftarrow$. Suppose that

$$\sum_{n=1}^{\infty} \left| \int_a^x f_n(t)\,dt \right|^2 \;=\; x-a, \qquad x \in [a,b].$$

Then,

$$\sum_{n=1}^{\infty} |\langle \chi_{[a,x]}, f_n\rangle|^2 \;=\; x-a \;=\; \|\chi_{[a,x]}\|_2^2.$$

Thus, the Plancherel Equality holds for $\chi_{[a,x]}$. This implies that $\chi_{[a,x]} \in \overline{\text{span}}\{f_n\}$. This is true for every $x \in [a,b]$, so

$$\chi_{[x,y]} \;=\; \chi_{[a,y]} - \chi_{[a,x]} \;\in\; \overline{\text{span}}\{f_n\}$$

for every $x < y$. The span of the set of characteristic functions of intervals, i.e.,

$$\text{span}\big\{\chi_{[x,y]} : a \le x < y \le b\big\},$$

is dense in $L^2[a,b]$ (this is sometimes called the set of "really simple functions"). Therefore $\overline{\text{span}}\{f_n\} = L^2[a,b]$.

6. Let S be a closed linear subspace of $L^1[0,1]$. Suppose that for each $f \in S$ there exists a $p > 1$ such that $f \in L^p[0,1]$. Show that there exists a $q > 1$ such that $S \subseteq L^q[0,1]$.

Solution: Since $S \subseteq L^1[0,1]$ and S is closed, it is complete since $L^1[0,1]$ is complete. Let q_n be real numbers that decrease to 1, and for m, $n \in \mathbb{N}$ set

$$E_{n,m} := \{f \in S : f \in L^{q_n}[0,1],\ \|f\|_{L^{q_n}} \leq m\}.$$

Then we have that

$$S = \bigcup_{n,m} E_{n,m},$$

since by hypothesis for each $f \in S$ there is a $p > 1$ such that $f \in L^p[0,1]$, and if $r > s$ then $L^r[0,1] \subseteq L^s[0,1]$, so choosing $q_n < p$ will place $f \in E_{n,m}$ for some m.

We claim that each set $E_{n,m}$ in the union is closed. To see this, suppose that $f_k \in E_{n,m}$ and $f_k \to f \in L^1[0,1]$. Since we are working in the topology defined by the L^1 norm, we can find a subsequence $\{f_{k_j}\}$ that converges to f almost everywhere on $[0,1]$. Then, as a consequence of Fatou's Lemma, we have

$$\int_0^1 |f|^{q_n}\,dx \ \leq\ \liminf_{j\to\infty} \int_0^1 \left|f_{k_j}\right|^{q_n} dx \leq m^{q_n}.$$

Hence $f \in E_{n,m}$, so $E_{n,m}$ is closed.

Therefore, by Baire's Category Theorem, there exist n_0, m_0 such that E_{n_0,m_0} has nonempty interior. So, there exists a ball of radius $\delta > 0$ centered at some point f such that

$$B_\delta(f) \subseteq E_{n_0,m_0}.$$

Let us assume that $f = 0$, as the general case can be handled similarly via a translation. Then we have that $B_\delta(0) \subseteq E_{n_0,m_o} \subseteq L^{q_{n_0}}[0,1]$.

Finally, choose any function $0 \neq g \in S$. Then

$$\widetilde{g} = \frac{\delta}{2}\frac{g}{\|g\|_{L^1}} \in B_\delta(0) \subseteq E_{n_0,m_o} \subseteq L^{q_{n_0}}[0,1].$$

But g is a scalar multiple of $\widetilde{g}$, so we conclude that $g \in L^{q_{n_0}}[0,1]$. This gives us $S \subseteq L^{q_{n_0}}[0,1]$.

7. Let k be a measurable function on $\mathbb{R}^2$ that satisfies

$$C_1 \;=\; \operatorname*{ess\,sup}_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |k(x,y)|\,dy \;<\; \infty,$$
$$C_2 \;=\; \operatorname*{ess\,sup}_{y\in\mathbb{R}} \int_{-\infty}^{\infty} |k(x,y)|\,dx \;<\; \infty.$$

Given $1 \le p \le \infty$, show that

$$L_k f(x) \;=\; \int_{-\infty}^{\infty} k(x,y)\,f(y)\,dy, \quad f \in L^p(\mathbb{R}),$$

defines a bounded mapping of $L^p(\mathbb{R})$ into itself, and its operator norm satisfies

$$\|L_k\|_{L^p \to L^p} \;\le\; C_1^{1/p'} C_2^{1/p}.$$

Solution: Suppose that $1 < p < \infty$ (the cases $p = 1$ and $p = \infty$ are similar). Given $f \in L^p(\mathbb{R})$,

$$\begin{aligned}
\|L_k f\|_p^p &= \int |L_k f(x)|^p\,dx \\
&= \int \left| \int k(x,y)\,f(y)\,dy \right|^p dx \\
&\le \int \left(\int |k(x,y)|^{1/p'} \cdot |k(x,y)|^{1/p}\,|f(y)|\,dy \right)^p dx \\
&\le \int \left(\int |k(x,y)|\,dy \right)^{p/p'} \left(\int |k(x,y)|\,|f(y)|^p\,dy \right) dx \\
&\le \int C_1^{p/p'} \int |k(x,y)|\,|f(y)|^p\,dy\,dx \\
&= C_1^{p/p'} \int |f(y)|^p \int |k(x,y)|\,dx\,dy \\
&\le C_1^{p/p'} \int |f(y)|^p\,C_2\,dy \\
&= C_1^{p/p'} C_2\,\|f\|_p^p.
\end{aligned}$$

Consequently,

$$\|L_k f\|_p \;\le\; C_1^{1/p'} C_2^{1/p}\,\|f\|_p,$$

so L_k is bounded and $\|L_k\|_{L^p \to L^p} \le C_1^{1/p'} C_2^{1/p}$.

8. Let X, Y, Z be Banach spaces. Suppose that $B\colon X \times Y \to Z$ is bilinear, i.e., $B_f(h) = B(f,h)$ and $B^g(h) = B(h,g)$ are linear functions of h for each $f \in X$ and $g \in Y$. Prove that the following three statements are equivalent.

(a) $B_f\colon Y \to Z$ and $B^g\colon X \to Z$ are continuous for each $f \in X$ and $g \in Y$.

(b) There is a constant $C > 0$ such that

$$\|B(f,g)\| \;\le\; C\,\|f\|\,\|g\|, \qquad f \in X,\; g \in Y.$$

(c) B is a continuous mapping of $X \times Y$ into Z (note that B need not be linear on the domain $X \times Y$).

Solution: (a) $\Rightarrow$ (b). Assume that B_f and B^g are continuous for each f and g.

Since B_f is bounded, for each individual $f \in X$ we have

$$\sup_{\|g\|=1} \|B^g(f)\| \;=\; \sup_{\|g\|=1} \|B_f(g)\| \;=\; \|B_f\| \;<\; \infty.$$

Since each operator B_g is linear, the Uniform Boundedness Principle therefore implies that

$$C \;=\; \sup_{\|g\|=1} \|B^g\| \;<\; \infty.$$

Now fix any vectors $f \in X$ and $g \in Y$. If $g \neq 0$ then $h = g/\|g\|$ is a unit vector in Y, so

$$\frac{1}{\|g\|}\,\|B(f,g)\| \;=\; \|B(f,h)\| \;=\; \|B^h(f)\| \;\le\; \|B^h\|\,\|f\| \;\le\; C\,\|f\|.$$

Therefore, we have shown that for all f and all nonzero g we have

$$\|B(f,g)\| \;\le\; C\,\|f\|\,\|g\|.$$

The inequality on the preceding line also holds trivially if $g = 0$, so statement (b) follows.

(b) $\Rightarrow$ (c). Assume that statement (b) holds. Suppose that $(f_n, g_n) \to (f,g)$ in $X \times Y$. Then $f_n \to f$ in X and $g_n \to g$ in Y, and consequently $D = \sup \|f_n\| < \infty$. Applying statement (b), it follows that

$$\begin{aligned}
\|B(f,g) - B(f_n,g_n)\| \;&\le\; \|B(f,g) - B(f_n,g)\| + \|B(f_n,g) - B(f_n,g_n)\| \\
&=\; \|B(f-f_n,g)\| + \|B(f_n,g-g_n)\| \\
&\le\; C\,\|f-f_n\|\,\|g\| + C\,\|f_n\|\,\|g-g_n\| \\
&\le\; C\,\|f-f_n\|\,\|g\| + CD\,\|g-g_n\| \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore B is continuous on $X \times Y$.

(c) $\Rightarrow$ (a). This follows immediately from the fact that convergence in $X \times Y$ implies convergence in each factor individually.