# Analysis Comprehensive Exam
# Fall 2013

**Instructions:** Complete 5 of the 8 problems below. If you attempt more than five questions, indicate clearly which five should be graded.

**Problem 1.** Let $|E|_e$ denote the exterior Lebesgue measure of a set $E \subset \mathbb{R}^n$. Suppose that $A$, $B$ are disjoint subsets of $[0,1]^n$ such that $A \cup B = [0,1]^n$. Show that $A$ and $B$ are measurable if and only if $|A|_e + |B|_e = 1$.

**Solution:** We denote by $|E|$ the Lebesgue measure of a measurable set $E$. Clearly, if $A$ and $B$ are measurable, then by the additivity of the Lebesgue measure we have $1 = |A| + |B| = |A|_e + |B|_e$.

Conversely, suppose that $1 = |A|_e + |B|_e$. Then, there exists a measurable set $U \supset A$ such that $|U| = |A|_e$, and by taking $U \cap [0,1]^n$ we can assume that $U \subset [0,1]^n$, i.e. $A \subset U \subset [0,1]^n$. Similarly, we can find a measurable set $V$ such that $B \subset V \subset [0,1]^n$ and $|B|_e = |V|$. Thus $V \supset B = [0,1]^n \setminus A \supset [0,1]^n \setminus U$, $U \supset A = [0,1]^n \setminus B \supset [0,1]^n \setminus V$ and $|U| + |V| = |A|_e + |B|_e = 1$. Then

$$|U \setminus A|_e \leq |U \setminus ([0,1]^n \setminus V)| = |U| - |[0,1]^n \setminus V| = 0,$$

hence $|U \setminus A|_e = 0$ and similarly $|V \setminus B|_e = 0$ which shows that $A$ and $B$ are measurable.

**Problem 2.** Let $(X, \rho)$ be a compact metric space and let $C(X)$ denote the space of continuous complex-valued functions on $X$ equipped with the uniform norm $||f||_u = \sup_{x \in X} |f(x)|$. Fix $\alpha > 0$ and for every $f \in C(X)$ let

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha}.$$

Show that $\mathcal{F} = \{f \in C(X) : ||f||_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

**Solution:** Clearly $\mathcal{F}$ is pointwise (even uniformly) bounded. Note also that, if $\rho(x,y) < \delta$ then for every $f \in \mathcal{F}$ we have $|f(x) - f(y)| \leq \rho(x,y)^\alpha < \delta^\alpha$, which shows that $\mathcal{F}$ is equicontinuous. Thus, by the Arzelà-Ascoli theorem we see that $\bar{\mathcal{F}}$ is compact. We prove next that $\mathcal{F}$ is closed (this would imply that $\mathcal{F} = \overline{\mathcal{F}}$ is compact, completing the proof).

Suppose that $\{f_n\}$ is a sequence in $\mathcal{F}$ and $||f_n - f||_u \to 0$. We want to show that $f \in \mathcal{F}$. The fact that the closed unit ball $\{g \in C(X) : ||g||_u \leq 1\}$ is closed in $C(X)$ implies that $||f||_u \leq 1$. To complete the proof it remains to show that $N_\alpha(f) \leq 1$.

Fix $x \neq y$ and let $\epsilon > 0$. Pick $n$ such that $||f_n - f||_u < \epsilon$. Then

$$\begin{aligned}\frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} &\leq \frac{|f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f(y) - f_n(y)|}{\rho(x,y)^\alpha} < \frac{2\epsilon + |f_n(x) - f_n(y)|}{\rho(x,y)^\alpha} \\ &\leq \frac{2\epsilon}{\rho(x,y)^\alpha} + N_\alpha(f_n) \leq \frac{2\epsilon}{\rho(x,y)^\alpha} + 1,\end{aligned}$$

because $f_n \in \mathcal{F}$. Since the above is true for every $\epsilon > 0$, we deduce that

$$\frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \leq 1.$$

Taking now a supremum over all $x \neq y$ we deduce that $N_\alpha(f) \leq 1$.

**Problem 3.** (a) Prove that a Banach space is either finite-dimensional or has uncountable dimension in the vector space sense (i.e. it is not generated by finite linear combinations of elements of some countable subset).
(b) Use part (a) to give an example of a vector space that cannot be given the structure of a Banach space.

**Solution:** (a) Let $X$ be a an infinite-dimensional Banach space, and suppose that $\{x_i : i \in \mathbb{N}\}$ is a countable basis for $X$ in the vector space sense (i.e. every element of $X$ is a finite linear combinations of $\{x_i\}$). Let $V_n = \text{span}\{x_1, x_2 \ldots, x_n\}$. Thus $X = \cup_{n=1}^{\infty} V_n$. Each $V_n$ is finite-dimensional and therefore closed. Note also that $V_n$ has an empty interior. Indeed, if $x \in V_n$ then $x + \epsilon x_{n+1} \notin V_n$ for any $\epsilon > 0$, hence $V_n$ contains no balls. Therefore, by the Baire Category Theorem $X \neq \cup_{n=1}^{\infty} V_n$ leading to a contradiction.
(b) We can take, for instance, $X$ to be the space of all infinite sequences with finitely many nonzero terms (which is isomorphic to the space of polynomials).

**Problem 4.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space.
(a) Show that every orthonormal sequence in $\mathcal{H}$ converges weakly to 0.
(b) Prove that the unit sphere $S = \{x : ||x|| = 1\}$ is weakly dense in the unit ball $B = \{x : ||x|| \leq 1\}$.

**Solution:** (a) Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathcal{H}$. In view of the Riesz representation theorem, we need to show that $\langle u_n, y\rangle \to 0$ for every $y \in \mathcal{H}$. This follows immediately from Bessel's inequality

$$\sum_{n\in\mathbb{N}} |\langle u_n, y\rangle|^2 \leq ||y||^2. \tag{1}$$

(b) Fix now $x \in B$ and let $\{v_\alpha\}_{\alpha\in A}$ be an orthonormal basis for $\mathcal{H}$. We construct below a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $S$ such that $x_n \to x$ weakly. We know that $\langle x, v_\alpha\rangle \neq 0$ for at most countably many $\alpha$'s, denote them by $\alpha_j$ where $j \in J$. Then we have

$$x = \sum_{j\in\mathbb{N}} \langle x, v_{\alpha_j}\rangle v_{\alpha_j},$$

where in the case when $J$ is finite, we have added countably many $\alpha_j$'s for which $\langle x, v_{\alpha_j}\rangle = 0$. From (1) we know that $\sum_{j\in\mathbb{N}} |\langle x, v_{\alpha_j}\rangle|^2 \leq 1$. If we set

$$x_n = \underbrace{\sum_{j=1}^{n} \langle x, v_{\alpha_j}\rangle v_{\alpha_j}}_{z_n} + \left(1 - \sum_{j=1}^{n} |\langle x, v_{\alpha_j}\rangle|^2\right)^{1/2} v_{\alpha_{n+1}}$$

then $x_n \in S$ by the Pythagorean theorem. Moreover, $z_n \to x$ with respect to the norm (hence weakly), $v_{\alpha_{n+1}} \to 0$ weakly by (a), and therefore $x_n \to x$ weakly.

**Problem 5.** Fix a measure space $(X, \mathcal{M}, \mu)$ and $1 \leq p < q < r < \infty$.
(a) Show that $L^p \cap L^r \subset L^q$ and $||f||_q \leq ||f||_p^\lambda \, ||f||_r^{1-\lambda}$, where $\lambda \in (0,1)$ is defined by $\lambda = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$.
(b) Prove that $L^p \cap L^r$ is a Banach space with norm $||f|| = ||f||_p + ||f||_r$, and the inclusion map $L^p \cap L^r \to L^q$ is continuous.

**Solution:** (a) Note that $p' = p/(\lambda q)$ and $q' = r/((1-\lambda)q)$ are conjugate exponents. Using the Hölder's inequality with $p'$ and $q'$ we find

$$||f||_q^q = \int_X |f|^{\lambda q} \, |f|^{(1-\lambda)q} d\mu \leq \left\| |f|^{\lambda q} \right\|_{p'} \left\| |f|^{(1-\lambda)q} \right\|_{q'} = ||f||_p^{\lambda q} ||f||_r^{(1-\lambda)q}.$$

Taking $q$th roots, we see that $||f||_q \leq ||f||_p^\lambda \, ||f||_r^{1-\lambda}$, completing the proof of (a).
(b) Since both $L^p$ and $L^r$ are vector spaces, it follows that $L^p \cap L^r$ is also a vector space. Moreover, using the fact that $|| \cdot ||_p$ and $|| \cdot ||_r$ are norms, it is easy to see that $|| \cdot ||$ is a norm. Indeed, by the triangle inequality (or Minkowski's inequality) for $|| \cdot ||_p$ and $|| \cdot ||_r$ we see that

$$||f + g|| = ||f + g||_p + ||f + g||_r \leq ||f||_p + ||g||_p + ||f||_r + ||g||_r = ||f|| + ||g||,$$

proving the triangle inequality for $|| \cdot ||$. Similarly, for every $c \in \mathbb{C}$ we have

$$||cf|| = ||cf||_p + ||cf||_r = |c| \, ||f||_p + |c| \, ||f||_r = |c|(||f||_p + ||f||_r) = |c| \, ||f||.$$

Finally, note that $||f|| = ||f||_p + ||f||_r = 0$ iff $||f||_p = ||f||_r = 0$ which is equivalent to $f = 0$ a.e. (since $|| \cdot ||_p$ and $|| \cdot ||_r$ are norms) and therefore $||f|| = 0$ iff $f = 0$ a.e.

Thus $L^p \cap L^r$ is a normed space and we need to prove that it is complete. Recall that if $f_n \to f$ in $L^p$ then there is a subsequence $f_{n_k} \to f$ a.e. Suppose now that $\{f_n\}$ is a Cauchy sequence with respect to $|| \cdot ||$. Since $||f_n - f_m|| = ||f_n - f_m||_p + ||f_n - f_m||_r$ we see that $\{f_n\}$ is a Cauchy sequence in $L^p$ and $L^r$ (which are Banach spaces, hence complete). Therefore there exist $f \in L^p$ and $g \in L^r$ such that $f_n \to f$ in $L^p$ and $f_n \to g$ in $L^r$. From the above remark, we can find a subsequence of $\{f_n\}$ which converges pointwise a.e. to $f$ and $g$, therefore $f = g$ a.e., i.e. $f = g$ as an element of $L^p \cap L^r$. From this we deduce that $||f_n - f|| = ||f_n - f||_p + ||f_n - f||_r \to 0$ proving the completeness.

Finally, since the inclusion map $L^p \cap L^r \to L^q$ is clearly linear, it is enough to show that it is bounded. But this follows immediately from (a) since

$$||f||_q \leq ||f||_p^\lambda \, ||f||_r^{1-\lambda} \leq ||f||^\lambda \, ||f||^{1-\lambda} = ||f||.$$

**Problem 6.** Let $E \subset \mathbf{R}^n$ be a measurable set, $f$ be a measurable function on $E$ and $\int_E f(x) \; dx = r > 0$, prove that: there exists a measurable subset $e \subset E$ such that

$$\int_e f(x) \; dx = \frac{r}{3}.$$

**Solution:** For $t \geq 0$, define the set $E_t = E \cap B(0;t)$, where $B(0;t)$ is the ball centered at origin with radius $t$. Let

$$F(t) = \int_{E_t} f(x)\ dx.$$

Then $F(t)$ is continuous on $[0,\infty)$. Since by the absolute continuity of Lebesgue integral, $\forall \varepsilon > 0$ and $\forall t \in [0,\infty)$, $\exists \delta > 0$ such that when $t' \in [0,\infty)$, $|t - t'| < \delta$,

$$\left|F(t) - F(t')\right| = \left|\int_{E_{t'} - E_t} f(x)\ dx\right| < \varepsilon.$$

We have

$$\lim_{t\to 0} F(t) = 0 \text{ and } \lim_{t\to\infty} F(t) = r > 0.$$

Thus by the intermediate value property of continuous functions, $\exists t_0 \in (0,\infty)$ such that $F(t_0) = r/3$. The conclusion is proved by setting $e = E \cap B(0;t_0)$.

**Problem 7.** Let $f(x)$ be an increasing function on $[a,b]$ and

$$\int_a^b f'(x)\ dx = f(b) - f(a).$$

Show that: $f(x)$ is absolutely continuous on $[a,b]$.

**Solution:** For any $x \in [a,b]$, we have

$$\int_a^x f'(t)\ dt \leq f(x) - f(a), \quad \int_x^b f'(t)\ dt \leq f(b) - f(x).$$

So

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(t)\ dt \\ &= \int_a^x f'(t)\ dt + \int_x^b f'(t)\ dt \\ &\leq (f(x) - f(a)) + (f(b) - f(x)) \\ &= f(b) - f(a). \end{aligned}$$

Thus the two inequalities above must be equality, that is,

$$\int_a^x f'(t)\ dt = f(x) - f(a), \quad \int_x^b f'(t)\ dt = f(b) - f(x).$$

So

$$f(x) = f(a) + \int_a^x f'(t)\ dt$$

which implies that $f(x)$ is absolutely continuous.

**Problem 8.** Suppose $f(x) \in L^2([0,1])$. Let

$$g(x) = \int_0^1 \frac{f(t)}{|x-t|^{\frac{1}{2}}} dt, \ 0 < x < 1.$$

Show that:

$$\|g\|_{L^2[0,1]} \le 2\sqrt{2} \|f\|_{L^2[0,1]}.$$

**Solution:** First,

$$\begin{aligned}
|g(x)| &= \left| \int_0^1 \frac{f(t)}{|x-t|^{\frac{1}{2}}} dt \right| \\
&\le \int_0^1 \frac{1}{|x-t|^{\frac{1}{4}}} \frac{|f(t)|}{|x-t|^{\frac{1}{4}}} dt \\
&\le \left( \int_0^1 |x-t|^{-\frac{1}{2}} \, dt \right)^{\frac{1}{2}} \left( \int_0^1 \frac{|f(t)|^2}{|x-t|^{\frac{1}{2}}} dt \right)^{\frac{1}{2}} \\
&\le \left(2\sqrt{2}\right)^{\frac{1}{2}} \left( \int_0^1 \frac{|f(t)|^2}{|x-t|^{\frac{1}{2}}} dt \right)^{\frac{1}{2}},
\end{aligned}$$

since

$$\begin{aligned}
\left( \int_0^1 |x-t|^{-\frac{1}{2}} \, dt \right)^{\frac{1}{2}} &= \left( \int_0^x |x-t|^{-\frac{1}{2}} \, dt + \int_x^1 |x-t|^{-\frac{1}{2}} \, dt \right)^{\frac{1}{2}} \\
&= \left(2\left(\sqrt{x} + \sqrt{1-x}\right)\right)^{\frac{1}{2}} \le \left(2\sqrt{2}\right)^{\frac{1}{2}}.
\end{aligned}$$

So

$$\begin{aligned}
\left( \int_0^1 |g(x)|^2 \, dx \right)^{\frac{1}{2}} &\le \left( 2\sqrt{2} \int_0^1 \int_0^1 \frac{|f(t)|^2}{|x-t|^{\frac{1}{2}}} dt dx \right)^{\frac{1}{2}} \\
&= \left( 2\sqrt{2} \int_0^1 |f(t)|^2 \int_0^1 \frac{1}{|x-t|^{\frac{1}{2}}} dx \ dt \right)^{\frac{1}{2}} \\
&\le \left( \left(2\sqrt{2}\right)^2 \int_0^1 |f(t)|^2 \, dt \right)^{\frac{1}{2}} \\
&= 2\sqrt{2} \left( \int_0^1 |f(t)|^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}$$