Analysis Comprehensive Exam Questions
Fall 2014

NOTE: All functions in this exam are (extended) real-valued.

1. Assume $1 \le p < \infty$. Given $f \in L^p[0,\infty]$, show that

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt = 0.$$

Solution: The case $p = 1$ follows directly. For $1 < p < \infty$, let

$$q = \frac{p}{p-1}$$

be the dual index to p, and observe that

$$\frac{1}{q} - 1 = \frac{p-1}{p} - 1 = -\frac{1}{p}.$$

Applying Hölder's inequality, we therefore obtain

$$\begin{aligned}
\left|\frac{1}{x}\int_0^x f(t)\,dt\right| &\le \frac{1}{x}\int_0^\infty \chi_{[0,x]}(t)\,|f(t)|\,dt \\
&\le \frac{1}{x}\,\|\chi_{[0,x]}\|_q\,\|f\|_p \\
&= x^{-1}\,x^{\frac{1}{q}}\,\|f\|_p \\
&= \frac{1}{x^{1/p}}\,\|f\|_p \\
&\to 0 \quad \text{as } x \to \infty.
\end{aligned}$$

2. Assume that f is a monotone increasing function on $[a, b]$. Prove that the following two statements are equivalent.

(a) f is absolutely continuous.

(b) For every absolutely continuous function g on $[a, b]$ and every $x \in [a, b]$,

$$\int_a^x f(t)\, g'(t)\, dt \; + \int_a^x f'(t)\, g(t)\, dt \;=\; f(x)g(x) - f(a)g(a).$$

Solution: The product $F = fg$ is absolutely continuous. Each of f and g is differentiable a.e., and at any point t where f and g are both differentiable, the product rule tells us that

$$F'(t) \;=\; f(t)g'(t) + f'(t)g(t).$$

Applying the Fundamental Theorem of Calculus to the absolutely continuous function F, if $x \in [a, b]$ then

$$\begin{aligned}\int_a^x f(t)\, g'(t)\, dt \; + \int_a^x f'(t)\, g(t)\, dt \;&=\; \int_a^x F'(t)\, dt \;=\; F(x) - F(a) \\ &=\; f(x)g(x) - f(a)g(a).\end{aligned}$$

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. Since f is monotone increasing on the closed interval $[a, b]$, we know that f is differentiable a.e., $f' \geq 0$ a.e., and

$$\int_a^b |f'| = \int_a^b f' \leq f(b) - f(a) < \infty.$$

Therefore f' is integrable. Let $g = 1$, the constant function. Then g is absolutely continuous, and $g' = 0$. Applying statement (b), it follows that for each $x \in [a, b]$ we have

$$\begin{aligned} f(x) - f(a) \;=\; f(x)g(x) - f(a)g(a) \;&=\; \int_a^x f(t)\, g'(t)\, dt \; + \int_a^x f'(t)\, g(t)\, dt \\ &=\; \int_a^x f'(t)\, dt.\end{aligned}$$

Therefore f is absolutely continuous on $[a, b]$.

3. For each $k \in \mathbb{N}$ let $x_k = (x_k(1), x_k(2), \dots)$ be a vector in ℓ^2. Assume that for each n we have

$$\lim_{k\to\infty} x_k(n) = 0.$$

Must it be true that x_k converges weakly to 0 as $k \to \infty$?

If we assume further that there exists some constant M such that $\|x_k\|_2 \le M$ for every k, must it then be true that x_k converges weakly to 0?

Solution: In the first case it need not be true that x_k converges weakly to 0. For example, let $x_k(n) = 1$ if $k \le n \le 2k$ and 0 otherwise. Then

$$\lim_{k\to\infty} x_k(n) = 0.$$

However, if we set $y = (y(1), y(2), \dots)$ with $y(n) = 1/n$, then $y \in \ell^2$ and the inner product of y with x_k satisfies

$$(y, x_k) = \sum_{n=k}^{2k} \frac{1}{n} \ge \int_k^{2k} \frac{1}{x}\,dx = \ln(2).$$

Hence x_k does not converge weakly to 0 as $k \to \infty$.

If we assume further that $\|x_k\|_2 \le M$ for every k, then the sequence $\{x_k\}_{k\in\mathbb{N}}$ is contained in the closed ball of radius M. Since ℓ^2 is a Hilbert space, closed balls are weakly compact. Therefore there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ that converges weakly, say to $y = (y(1), y(2), \dots)$. If we let δ_m denote the vector in ℓ^2 that has a 1 in the mth component and zeros elsewhere, then

$$y(m) = (y, \delta_m) = \lim_{n\to\infty} (x_{k_n}, \delta_m) = \lim_{n\to\infty} x_{k_n}(m) = 0.$$

Thus $y = 0$, so x_{k_n} converges weakly to the zero in ℓ^2.

We can replace $\{x_k\}_{n\in\mathbb{N}}$ in the preceding argument with any subsequence of $\{x_k\}_{n\in\mathbb{N}}$. This shows that every subsequence of $\{x_k\}_{n\in\mathbb{N}}$ contains a sub-subsequence that converges weakly to 0. This implies that the original sequence $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to 0.

Remark: The Uniform Boundedness Principle can be used to give an alternative proof.

4. Given $f \in L^1(\mathbb{R})$, define

$$F(x) \;=\; \int_{-\infty}^{\infty} f(t)\,\frac{\sin xt}{t}\,dt, \qquad x \in \mathbb{R}.$$

Prove that F is differentiable on $\mathbb{R}$, and it is absolutely continuous on any finite interval $[a,b]$.

Solution: $F(x)$ is well-defined for every x since f is integrable and $\frac{\sin xt}{t}$ is a bounded function of t. The issue is to pass the derivative under the integral; we justify this via the Dominated Convergence Theorem.

Let x be fixed, and set

$$S(t) = \sin t \qquad \text{and} \qquad g_x(t) = f(t)\,\frac{\sin xt}{t}.$$

Since S is differentiable, the Mean Value Theorem implies that given t and h, there exists some point ξ (depending on both t and h) such that

$$\frac{S(xt+ht) - S(xt)}{ht} = S'(\xi) = \cos \xi.$$

Hence

$$\left|\frac{g_{x+h}(t) - g_x(t)}{h}\right| \;=\; \left|f(t)\,\frac{S(xt+ht)-S(xt)}{ht}\right| \;=\; |f(t)\,S'(\xi)| \;\le\; |f(t)| \;\in\; L^1(\mathbb{R}).$$

Also, since S is differentiable,

$$\lim_{h\to 0}\frac{S(xt+ht) - S(xt)}{ht} = S'(xt) = \cos xt.$$

Therefore

$$\lim_{h\to 0}\frac{g_{x+h}(t) - g_x(t)}{h} \;=\; \lim_{h\to 0} f(t)\,\frac{S(xt+ht)-S(xt)}{ht} \;=\; f(t)\,\cos xt.$$

Applying the Dominated Convergence Theorem, $F'(x)$ exists and has the value

$$F'(x) \;=\; \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \;=\; \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{g_{x+h}(t)-g_x(t)}{h}\,dt \;=\; \int_{-\infty}^{\infty} f(t)\,\cos xt\,dt.$$

Since

$$|F'(x)| \le \int_{-\infty}^{\infty} |f(t)\,\cos xt|\,dt \le \int_{-\infty}^{\infty} |f(t)|\,dt \;=\; \|f\|_1,$$

we see that F' is bounded, and therefore F is Lipschitz and hence absolutely continuous on any interval $[a,b]$.

5. Let μ be a finite signed Borel measure on $\mathbb{R}$, and suppose that μ is absolutely continuous with respect to Lebesgue measure. Show that if A is a Borel measurable subset of $\mathbb{R}$, then the function $\mu(A+t)$ is continuous in t (here $A+t=\{x+t : x\in A\}$).

Solution: Since μ is absolutely continuous with respect to Lebesgue measure, there exists an extended real-valued, Borel measurable function f such that for every Borel set A we have

$$\mu(A) = \int_A f(x)\,dx.$$

If we let $\mathbb{R} = P \cup N$ be a Hahn decomposition of $\mathbb{R}$ for μ, then μ is nonnegative on P and nonpositive on N. Consequently, f is nonnegative a.e. on P and nonpositive a.e. on N. As $0 \le \mu(P) < \infty$ and $0 \le -\mu(N) < \infty$, it follows that

$$\int_{\mathbb{R}} |f| = \int_P f - \int_N f = \mu(P) - \mu(N) < \infty.$$

Thus f is integrable.

Fix $t_0 \in \mathbb{R}$. If we set $T_t f(x) = f(x-t)$, then by standard methods (approximation by a continuous, compactly supported function), we have

$$\lim_{t\to t_0} \|T_t f - T_{t_0} f\|_1 = 0.$$

Therefore, given a Borel set A,

$$\begin{aligned}
\lim_{t\to t_0} |\mu(A+t) - \mu(A+t_0)| &= \lim_{t\to t_0} \left| \int_{A+t} f(x)\,dx - \int_{A+t_0} f(x)\,dx \right| \\
&= \lim_{t\to t_0} \left| \int_A f(x-t)\,dx - \int_A f(x-t_0)\,dx \right| \\
&\le \lim_{t\to t_0} \int_A |T_t f(x) - T_{t_0} f(x)|\,dx \\
&\le \lim_{t\to t_0} \|T_t f - T_{t_0} f\|_1 \\
&= 0.
\end{aligned}$$

Therefore $\mu(A+t)$ is continuous at t_0.

An alternative proof is to proceed through cases from $A = (a,b)$ to A being an arbitrary Borel set.

6. Let $C_c(\mathbb{R})$ denote the space of continuous, compactly supported functions on $\mathbb{R}$, and let $C_0(\mathbb{R})$ denote the set of all continuous functions f such that $\lim_{|x|\to\infty} f(x) = 0$. Prove that $C_c(\mathbb{R})$ is a meager (= 1st category) subset of $C_0(\mathbb{R})$.

Solution: For each $N \in \mathbb{N}$, define

$$C[-N,N] = \big\{ f \in C(\mathbb{R}) : \operatorname{supp}(f) \subseteq [-N,N] \big\}.$$

Since uniform convergence implies pointwise convergence, this is a closed subspace of $C_0(\mathbb{R})$ with respect to the uniform norm.

Given $\varepsilon > 0$, let g be any nonzero continuous function supported on $[N, N+1]$ such that $\|g\|_\infty < \varepsilon$. Given any $f \in C[-N,N]$ we have $h = f + g \notin C[-N,N]$ yet

$$\|f - h\|_\infty = \|g\|_\infty < \varepsilon.$$

Therefore $h \in B_\varepsilon(f)$, so $B_\varepsilon(f)$ is not contained in $C[-N,N]$. Thus $C[-N,N]$ is a closed subset of $C_0(\mathbb{R})$ that has no interior, which says that $C[-N,N]$ is a nowhere dense subset of $C_0(\mathbb{R})$. Finally,

$$C_c(\mathbb{R}) = \bigcup_{N\in\mathbb{N}} C[-N,N],$$

so $C_c(\mathbb{R})$ is a meager subset of $C_0(\mathbb{R})$.

7. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Banach space X such that $\sum_{n=1}^{\infty} |\mu(x_n)|$ converges uniformly for μ in the unit sphere in the dual space X^*, i.e.,

$$\lim_{N\to\infty} \sup_{\mu\in X^*,\, \|\mu\|=1} \sum_{n=N}^{\infty} |\mu(x_n)| = 0.$$

Prove that $\sum_{n=1}^{\infty} c_n x_n$ converges for every bounded sequence of scalars $(c_n)_{n\in\mathbb{N}}$.

Hint: Prove that $\sum c_n x_n$ is a Cauchy series.

Solution: Choose any bounded sequence $c = (c_n)_{n\in\mathbb{N}}$, and fix $\varepsilon > 0$. The result is trivial if c is the zero sequence, so assume that not every c_n is zero. By hypothesis, there exists an N_0 such that if $N > N_0$ then

$$\sup_{\|\mu\|=1} \sum_{n=N}^{\infty} |\mu(x_n)| < \frac{\varepsilon}{\|c\|_\infty}.$$

Consider any $N_0 < M < N$. Then by Hahn–Banach we have

$$\begin{aligned}
\left\| \sum_{k=1}^{N} c_n x_n - \sum_{k=1}^{M} c_n x_n \right\| &= \left\| \sum_{k=M+1}^{N} c_n x_n \right\| \\
&= \sup_{\|\mu\|=1} \left| \mu\Big(\sum_{k=M+1}^{N} c_n x_n \Big) \right| \\
&= \sup_{\|\mu\|=1} \left| \sum_{k=M+1}^{N} c_n \mu(x_n) \right| \qquad \text{(linearity)} \\
&\le \sup_{\|\mu\|=1} \sum_{k=M+1}^{N} |c_n|\, |\mu(x_n)| \\
&\le \|c\|_\infty \sup_{\|\mu\|=1} \sum_{k=M+1}^{N} |\mu(x_n)| \\
&< \varepsilon.
\end{aligned}$$

Hence the sequence of partial sums $\left\{\sum_{k=1}^{N} c_n x_n\right\}_{N\in\mathbb{N}}$ is Cauchy in X, and must therefore converge since X is a Banach space. This tells us that the infinite series $\sum_{k=1}^{\infty} c_n x_n$ converges.

8. Let μ^* be an outer measure on a set X, and let $A_1, A_2, \ldots$ be disjoint μ^*-measurable subsets of X. Prove that for any $E \subset X$ (measurable or not),

$$\mu^*\Big(E \cap \Big(\bigcup_{j=1}^{\infty} A_j\Big)\Big) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Solution: First we will prove by induction that if $n \in \mathbb{N}$, then

$$\mu^*\Big(E \cap \Big(\bigcup_{j=1}^{n} A_j\Big)\Big) = \sum_{j=1}^{n} \mu^*(E \cap A_j). \tag{1}$$

This statement is trivially true if $n = 1$. Suppose that equation (1) holds for some n, and let $B_n = \bigcup_{j=1}^{n} A_j$. Applying Carathéodory's Criterion, we compute that

$$\begin{aligned}
\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap A_{n+1}) + \mu^*(E \cap B_{n+1} \cap A_{n+1}^{\mathrm{C}}) && (A_{n+1} \text{ is measurable})\\
&= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap B_n) && (\text{disjointness of the } A_j)\\
&= \mu^*(E \cap A_{n+1}) + \sum_{j=1}^{n} \mu^*(E \cap A_j)\\
&= \sum_{j=1}^{n+1} \mu^*(E \cap A_j).
\end{aligned}$$

This establishes equation (1).

Finally, taking B_n as before and setting $B = \bigcup_{j=1}^{\infty} A_j$, we use the above work, monotonicity, and subadditivity to compute that

$$\sum_{j=1}^{n} \mu^*(E \cap A_j) = \mu^*(E \cap B_n) \leq \mu^*(E \cap B) = \mu^*\Big(\bigcup_{j=1}^{\infty} (E \cap A_j)\Big) \leq \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

This holds for every n, and every term in the sum is nonnegative, so we can take the limit as $n \to \infty$ and the result follows.