Algebra Comprehensive Exam
Aug 28, 2015

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Please note that a complete solution of a problem is preferable to partial progress on several problems. Write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Let $\mathbf{F}$ be a finite field and let M be an invertible $n \times n$ matrix with entries in $\mathbf{F}$. Prove that $M^m - I_n$ is not invertible for some integer $m \geq 1$. (I_n denotes the $n \times n$ identity matrix.)

   Alternate:

2. Let G be a non-abelian finite group with center $Z(G)$. Prove that $\#Z(G) \leq \frac{1}{4}\#G$.

3. Which of the following rings are isomorphic? Justify your answer.

   (a) $R_1 = \mathbf{Q}[X]/(X^2 - 1)$

   (b) $R_2 = \mathbf{Q}[X]/(X^2 - 2)$

   (c) $R_3 = \mathbf{Q}[X]/(X^2 - 3)$

   (d) $R_4 = \mathbf{Q}[X]/(X^2 - 4)$

   Alternate:

4. (a) Give an example of a degree-6 Galois extension $F/\mathbf{Q}$ with non-abelian Galois group.

   (b) Give an example of a degree-6 Galois extension $K/\mathbf{Q}$ with abelian Galois group.

   Justify your answers.

5. For a linear operator $A\colon V \to V$ on a finite-dimensional real vector space V, such that $A^2 = A$, show that $\operatorname{trace} A = \operatorname{rank} A$.

   Alternate:

6. Let G be an abelian group with generators a, b, c and relations

$$M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \text{ where } M = \begin{bmatrix} 2 & 10 & 6 \\ 4 & 6 & 12 \\ 2 & 4 & 6 \end{bmatrix}.$$

   (a) Find the decomposition of G according to the Fundamental Theorem of finitely generated abelian groups.

   (b) What are cyclic generators corresponding to the components in this decomposition in terms of a, b, c?

7. Show that for any field F and any integer $d \geq 1$ there exists at most one finite multiplicative subgroup $G \subset F \setminus \{0\}$ of order d.

8. Let R be a commutative ring and let $f(X) = \sum_{i=0}^{d} c_i X^i$ be a *nilpotent* univariate polynomial with coefficients $c_i \in R$. Show that the coefficients c_i are also nilpotent.