Analysis Comprehensive Exam
August 21, 2015

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Please note that a complete solution of a problem is preferable to partial progress on several problems. Write only on the front side of the solution pages. Work on the back of the page might not be graded.
1. Let \( f \in L^1(\mathbb{R}) \) and \( \alpha > 0 \). Show that 
\[
\lim_{n \to \infty} f(nx)n^{-\alpha} = 0 \quad \text{for a.e. } x \in \mathbb{R}.
\]

2. Suppose that \( f_n \) are absolutely continuous functions on \([0, 1]\) such that \( f_n(0) = 0 \) and 
\[
\sum_{n=1}^{\infty} \int_{0}^{1} |f'_n(x)| \, dx < \infty.
\]
show that

- \( \sum_{n=1}^{\infty} f_n(x) \) converges for every \( x \). Call the limit \( f(x) \);
- \( f \) is absolutely continuous;
- for a.e. \( x \in [0, 1] \), we have 
  \[ f'(x) = \sum_{n=1}^{\infty} f'_n(x). \]

3. Let \( f_n \) be a sequence in \( L^2([0, 1]) \) and, for \( x \in [0, 1] \), define 
\[
F_n(x) = \int_{0}^{x} f_n(t) \, dt
\]
Assume that \( f_n \) converge in norm to \( f \) in \( L^2([0, 1]) \) with 
\[
F(x) = \int_{0}^{x} f(t) \, dt
\]
a) Show that \( F_n, F \) are continuous and that \( F_n \) converge to \( F \) uniformly on \([0, 1]\).
b) Is the conclusion still true if \( f_n \) converge weakly to \( f \)? Prove or find a counterexample.

4. Given two measurable sets \( A \) and \( B \) in \( S^1 = \mathbb{R}/\mathbb{Z} \) let
\[
\tau_y(A) = (A + y) \mod 1.
\]
Let \( m \) be the Lebesgue measure on \( S^1 \) and note that \( m(S^1) = 1 \). Show that there exists \( y \in S^1 \) such that 
\[
m(\tau_y(A) \cap B) \geq m(A)m(B).
\]

5. Show that every closed convex set \( K \) in a Hilbert space \( \mathcal{H} \) has a unique element of minimal norm.
6. Let \((X, \mathcal{M}, \mu)\) be a finite measure space, \(\mathcal{N}\) a sub-\(\sigma\)-algebra of \(\mathcal{M}\) and \(\nu = \mu|_\mathcal{N}\) be the restriction of \(\mu\) to \(\mathcal{N}\). Show that for every \(f \in L^1(X, \mathcal{M}, \mu)\) there exists a unique \(P(f) \in L^1(X, \mathcal{N}, \nu)\) such that \(\int_E f \, d\mu = \int_E P(f) \, d\nu\) for all \(E \in \mathcal{N}\). Moreover, \(L^1(X, \mathcal{N}, \nu)\) is a closed linear subspace of \(L^1(X, \mathcal{M}, \mu)\) and \(P\) is a continuous linear projection (i.e. \(P^2 = P\)) of \(L^1(X, \mathcal{M}, \mu)\) onto \(L^1(X, \mathcal{N}, \nu)\).

7. Fix a finite measure space \((X, \mathcal{M}, \mu)\) and \(1 \leq p < q \leq \infty\). Show that \(L^p \not\subset L^q\) iff \(X\) contains sets of arbitrarily small positive measure (Hint for the “if” implication: construct a disjoint sequence \(\{E_n\}\) with \(0 < \mu(E_n) < \frac{1}{2^n}\), and consider \(f = \sum_n a_n \chi_{E_n}\) for suitable constants \(a_n\).).

8. Let \(1 < p < \infty\). Show that the operator \(Tf(x) = \int_0^\infty \frac{f(y)}{x+y} \, dy\) satisfies
\[
\|Tf\|_p \leq C_p \|f\|_p,
\]
where \(C_p = \int_0^\infty \frac{dx}{(1 + x)x^{1/p}}\), and \(\| \cdot \|_p\) is the \(p\)-norm on \(L^p(0, \infty)\).