Analysis Comprehensive Exam
August 21, 2015

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Please note that a complete solution of a problem is preferable to partial progress on several problems. Write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Let $f \in L^1(\mathbb{R})$ and $\alpha > 0$. Show that

$$\lim_{n\to\infty} f(nx)n^{-\alpha} = 0 \qquad \text{for a.e. } x \in \mathbb{R}.$$

2. Suppose that f_n are absolutely continuous functions on $[0,1]$ such that $f_n(0) = 0$ and

$$\sum_{n=1}^{\infty} \int_0^1 |f_n'(x)|dx < \infty.$$

show that

- $\sum_{n=1}^{\infty} f_n(x)$ converges for every x. Call the limit $f(x)$;
- f is absolutely continuous;
- for a.e. $x \in [0,1]$, we have

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

3. Let f_n be a sequance in $L^2([0,1])$ and, for $x \in [0,1]$, define

$$F_n(x) = \int_0^x f_n(t)dt$$

Assume that f_n converge in norm to f in $L^2([0,1])$ with

$$F(x) = \int_0^x f(t)dt$$

a) Show that F_n, F are continuous and that F_n converge to F uniformly on $[0,1]$.

b) Is the conclusion still true if f_n converge weakly to f? Prove or find a counterexample.

4. Given two measurable sets A and B in $S^1 = \mathbb{R}/\mathbb{Z}$ let

$$\tau_y(A) = (A + y) \bmod 1.$$

Let m be the Lebesgue measure on S^1 and note that $m(S^1) = 1$. Show that there exists $y \in S^1$ such that

$$m(\tau_y(A) \cap B) \geq m(A)m(B).$$

5. Show that every closed convex set K in a Hilbert space $\mathcal{H}$ has a unique element of minimal norm.

6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$ and $\nu = \mu|_{\mathcal{N}}$ be the restriction of μ to $\mathcal{N}$. Show that for every $f \in L^1(X, \mathcal{M}, \mu)$ there exists a unique $P(f) \in L^1(X, \mathcal{N}, \nu)$ such that $\int_E f\, d\mu = \int_E P(f)\, d\nu$ for all $E \in \mathcal{N}$. Moreover, $L^1(X, \mathcal{N}, \nu)$ is a closed linear subspace of $L^1(X, \mathcal{M}, \mu)$ and P is a continuous linear projection (i.e. $P^2 = P$) of $L^1(X, \mathcal{M}, \mu)$ onto $L^1(X, \mathcal{N}, \nu)$.

7. Fix a finite measure space $(X, \mathcal{M}, \mu)$ and $1 \le p < q \le \infty$. Show that $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure (Hint for the "if" implication: construct a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) < \frac{1}{2^n}$, and consider $f = \sum_n a_n \chi_{E_n}$ for suitable constants a_n.).

8. Let $1 < p < \infty$. Show that the operator $Tf(x) = \int_0^\infty \frac{f(y)}{x+y}\, dy$ satisfies

$$||Tf||_p \le C_p ||f||_p, \qquad \text{where } C_p = \int_0^\infty \frac{dx}{(1+x)x^{1/p}},$$

and $||\cdot||_p$ is the p-norm on $L^p(0, \infty)$.