Numerical Analysis Comprehensive Exam
Aug 26, 2015

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers exactly in the box below—the uncircled problems will **not** be graded. At least 3 of the 5 problems selected must be from the last 4 problems.

1	2	3	4	5	6	7	8

Please note that a complete solution of a problem is preferable to partial progress on several problems. Write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Find a_0, a_1 and x_1 so that the following numerical integration formula

$$\int_0^h f(x)\,dx = a_0 f(0) + a_1 f(x_1) + R$$

has the highest polynomial degree of precision. Here $R = O(h^r)$ denotes the order of approximation error. What is r ?

2. Find a formula for the following polynomial interpolation problem. Let $x_i = x_0 + ih$, $i = 0, 1, 2, 3$, $h > 0$. Find a polynomial $p(x)$ of degree ≤ 5 for which

$$p(x_i) = f(x_i), \quad i = 0, 1, 2, 3;$$
$$p'(x_0) = f'(x_0), \quad p''(x_0) = f''(x_0),$$

where $f(x)$ has continuous derivative of any order in $(-\infty, \infty)$. Derive an error formula for $f(x) - p(x)$. What's the order of approximation for $x \in [x_0, x_3]$?

3. The following two iterative schemes are designed to compute $\sqrt{a}$ where $a > 0$ is a constant.
Scheme A:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}).$$

Scheme B:

$$x_{n+1} = x_n + x_n^2 - a.$$

Are the schemes convergent? If so, find out the rate of convergence. You must justify your answers.

4. To compute the solution of an ODE $y' = f(t, y)$ with initial condition $y(0) = y_0$ numerically, the following scheme is used,

$$y_{n+1} = 3y_n - 2y_{n-1} + \frac{h}{2}[f(t_n, y_n) - 3f(t_{n-1}, y_{n-1})].$$

- Is this scheme consistent? If so, what is the order of the scheme. If not, show your reason.
- Is this scheme useful in practice? You must justify your answer.

5. Assume that $\vec{x}$ is sufficiently close to an eigenvector $\vec{q}$ of a symmetric matrix A with corresponding eigenvalue λ. Show that

$$r(\vec{x}) = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$$

is an approximation to λ. Use $r(\vec{x})$ to design an algorithm to compute $\vec{q}$ and λ with at least second order convergence rate. You must justify your claim.

6. Consider a linear system of equations

$$A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}.$$

   (a) Find all values of α such that A is symmetric positive definite.

   (b) Find all the values of α such that the Jacobi iteration is convergent when solving the linear system.

   (c) Final all the values of α such as the Gauss-Seidel iteration is convergent when solving the linear system.

   You must justify your answers.

7. Consider the equation $u_t + au_x = \gamma u_{xx}$ with $x \in [x_0, x_1]$, where $a > 0$ and $\gamma > 0$ are constants. Design a numerical scheme which is unconditionally stable for solving the equation with initial value and Dirichlet boundary conditions, and prove your claim.

8. For the two-point boundary value problem $-u_{xx}+u = f(x)$ on $[a, b]$ with $u(a) = u(b) = 0$, design a piecewise linear continuous finite element method for solving it, find a functional defined on the same finite element space whose minimizer is identical to the numerical solution of the finite element method and justify your answer.