Topology Comprehensive Exam
August 14, 2015

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below — the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Please note that a complete solution of a problem is preferable to partial progress on several problems. Write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Let M be a smooth compact manifold of dimension n (without boundary) and C a submanifold of M diffeomorphic to a circle. If there is a smooth function $f : M \to S^1$ that is a diffeomorphism when restricted to C then show that there is a $(n-1)$-dimensional submanifold of M that is not the boundary of any compact submanifold in M.

2. Let M be a compact m-dimensional manifold and N be an n-dimensional manifold. Suppose that $f : M \to N$ is a bijection and $df_x : T_xM \to T_{f(x)}N$ is injective at each $x \in M$. Show that f is a diffeomorphism.

3. Given an area form ω on a surface Σ (that is a 2–form that is never zero) then one can define the divergence of a vector field v on Σ as the unique function $\mathrm{div}_\omega v$ such that

$$\mathcal{L}_v\omega = (\mathrm{div}_\omega v)\omega,$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to v.

   (a) Show that if ω' is another area form (defining the same orientation) then there is a unique positive function f such that $\omega' = f\omega$ and that

$$\mathrm{div}_\omega(v) = \mathrm{div}_{\omega'}(v) + d(\ln f)(v).$$

   (b) Derive a formula for $\mathrm{div}_\omega(v')$ in terms of $\mathrm{div}_\omega(v)$ if $v' = gv$ for some function g.

   (c) Show that given a function $f : \Sigma \to \mathbb{R}$ there is a unique vector field v_f that satisfies $\iota_{v_f}\omega = df$, where $\iota_{v_f}\omega$ is the contraction of ω given by $\iota_{v_f}\omega(x) = \omega(v_f, x)$.

   (d) Show the flow of v_f from the previous item preserves the level sets of f and has zero divergence.

4. Let $X = S^2 - \{x_0, \ldots, x_n\}$ be the 2-sphere with $n+1$ distinct points removed. Choose a base-point b such that a geodesic path $\wp_i$ from b to each x_i doesn't go through any other x_i. Let B_i be a small closed ball around x_i, so that all the B_i are disjoint and don't intersect any $\wp_j$ for $j \neq i$. Let γ_i be the path from b following $\wp_i$ to the boundary B_i, then following the boundary of B_i counterclockwise, and then returning along $\wp_i$.

   (a) Prove that there is a unique homomorphism $f : \pi_1(X, b) \to \mathbb{Z}/2$ sending γ_i to 1 for all i if and only if n is odd.

   (b) By the classification of covering spaces, there is a unique covering space $Y \to X$ corresponding to $\mathrm{Ker}\, f$. Compute the abelianization of $\pi_1(Y)$ as a finitely generated abelian group.

5. Let X be the topological space

$$X = \{(x,y) \in \mathbb{C}^2\} - \Big(\{(x,y) : x = y\} \cup \{(x,y) : x = -y\} \cup \{(x,y) : x = y+1\} \cup \{(x,y) : x = -y+1\}\Big).$$

Show that any map from $\mathbb{RP}^3 \to X$ is null-homotopic.

6. The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times [0,1]$ obtained by identifying $(x, 0)$ with $(f(x), 1)$. Let $X = S^1 \vee S^1$ and let $f : X \to X$ be the following map. View S^1 as the subset of elements of $\mathbb{C}$ of elements of norm 1, with base point 1. Let f map the first S^1 to the second by $z \mapsto z^2$, and let f map the second to the first by $z \mapsto z^{-3}$. Give a presentation of $\pi_1(T_f)$.

7. Let X be a path-connected, locally path-connected, and semi locally simply-connected topological space. A covering space $f : Y \to X$ is said to be *finite* if $f^{-1}(x)$ is a finite set for all $x \in X$. A covering space $f : Y \to X$ is said to be *Galois* if for any points $y_1, y_2 \in Y$ such that $f(y_1) = f(y_2)$, there exists a covering transformation $g : Y \to Y$ such that $g(y_1) = y_2$.

   Show that for any connected finite covering space $Y \to X$ there exists a finite Galois covering space $Z \to X$ such that $Z \to X$ factors $Z \to Y \to X$.

8. Let $f : S^1 \to \mathbb{R}^3$ be a knot, i.e., a smooth embedding. Given an element $v \in S^2$, let $P_v = \{v\}^\perp$ denote the plane perpendicular to v and let $\pi_v : \mathbb{R}^3 \to P_v$ denote orthogonal projection.

   (a) Show that for almost every $v \in S^2$, $f_v = \pi_v \circ f : S^1 \to P_v$ is an immersion.

   (b) Let $X = S^1 \times S^1 - \Delta$ where $\Delta = \{(x, x) : x \in S^1\}$ is the diagonal. Consider the map $G : X \to S^2$ defined by

$$G(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.$$

   Show that if $v \in S^2$ is a regular value of G and f_v is an immersion, then f_v has transverse crossings in the sense that $f_v(x) = f_v(y)$ implies that $\partial f_v(x)$ and $\partial f_v(y)$ are linearly independent. Conclude that for almost all $v \in S^2$, f_v is an immersion with transverse crossings.