Attempt any five questions, and please provide careful and complete answers. If you attempt more than five questions, specify which should be graded.

1. How many elements of the permutation group S_7 commute with the permutation $(12)(34) \in S_7$?

   Solution: Consider the conjugation action of S_7 on itself. The orbit X of $(12)(34)$ consists of all elements of the form $(ab)(cd)$ where a, b, c, d are distinct, and so the cardinality of the orbit is

   $$|X| = \frac{1}{2}\binom{7}{2}\binom{5}{2}.$$

   Therefore the number of elements of S_7 which commute with $(12)(34)$ is $\dfrac{|S_7|}{|X|} = 48$.

2. Recall that $SL_2(\mathbb{Z}/5\mathbb{Z})$ is the group of 2×2 matrices of determinant 1, which have entries in $\mathbb{Z}/5\mathbb{Z}$. Determine the number of 5-Sylow subgroups of $SL_2(\mathbb{Z}/5\mathbb{Z})$.

   Solution: Let p be any prime. A counting argument shows that

   $$|SL_2(\mathbb{Z}/p\mathbb{Z})| = p(p-1)(p+1),$$

   so a p-Sylow subgroup has p elements. The number of p-Sylow subgroups is $1 \mod p$, and divides $|SL_2(\mathbb{Z}/p\mathbb{Z})|/p = (p-1)(p+1)$.

   When $p = 5$, this shows that the number of 5-Sylow subgroups is either 1 or 6. However each of

   $$P = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}/5\mathbb{Z} \right\} \qquad \text{and} \qquad Q = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{Z}/5\mathbb{Z} \right\}$$

   is a 5-Sylow subgroup, so there must be exactly six 5-Sylow subgroups.

   Alternately, one may use matrix multiplication to compute the normalizer of a 5-Sylow subgroup, say of P above, in which case we get

   $$N_P = \left\{ M \in SL_2(\mathbb{Z}/5\mathbb{Z}) : MPM^{-1} = P \right\} = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} : a, b \in \mathbb{Z}/5\mathbb{Z}, a \neq 0 \right\}.$$

   The number of 5-Sylow subgroups is $\dfrac{|SL_2(\mathbb{Z}/5\mathbb{Z})|}{|N_P|} = 6$.

3. Let I be the ideal of the polynomial ring $\mathbb{R}[x]$ generated by the elements $x^3 + 1$ and $x^4 + 3x + 2$. Is it possible to generate I by one element? Is I a prime ideal? Is it maximal?

   Solution: $\mathbb{R}[x]$ is a principal ideal domain, so I must be generated by one element and, indeed,

   $$I = (x^3 + 1, x^4 + 3x + 2) = (x^3 + 1, 2x + 2) = (x + 1).$$

   Since $x + 1$ is an irreducible polynomial, the ideal I is prime and also maximal.

4. Let K be a field with 49 elements. For each of the polynomials below, determine the number of distinct roots in K.

$$x^{48} - 1, \qquad x^{49} - 1, \qquad x^{54} - 1.$$

Solution: The nonzero elements of K form a multiplicative group of order 48, hence these 48 elements are precisely the roots of $x^{48} - 1$.

Since K has characteristic 7 we get $x^{49} - 1 = (x-1)^{49}$, which has one distinct root—the element $1 \in K$.

Since nonzero elements of K satisfy $x^{48} - 1$, the roots of $x^{54} - 1$ in K are precisely the roots of $x^6 - 1$. There are 6 such roots in K.

5. Let $\xi = e^{2\pi i/5}$. Determine the degrees of the field extensions

(a) $[\mathbb{Q}(\xi) : \mathbb{Q}]$,

(b) $[\mathbb{Q}(\xi, \sqrt{5}) : \mathbb{Q}]$.

Solution: (a) ξ is a root of the irreducible polynomial $x^4 + x^3 + x^2 + x + 1$, hence $[\mathbb{Q}(\xi) : \mathbb{Q}] = 4$.

(b) The issue is whether the field $\mathbb{Q}(\xi)$ contains the real number $\sqrt{5}$. Note that $\xi + 1/\xi$ is a root of $x^2 + x - 1$, and so $\xi + 1/\xi = (\sqrt{5} - 1)/2$. Hence $\sqrt{5} \in \mathbb{Q}(\xi)$, and so $[\mathbb{Q}(\xi, \sqrt{5}) : \mathbb{Q}]$=4.

6. Determine all 3×3 matrices M with rational entries for which M^7 is the identity matrix.

Solution: If $M^7 = I$, the minimal polynomial of M is a monic polynomial $f(x)$ in $\mathbb{Q}[x]$ which divides $x^7 - 1$. Since the factorization of $x^7 - 1$ in $\mathbb{Q}[x]$ is

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1),$$

the only possibility is $f(x) = x - 1$, and so M must be the identity matrix.

7. (a) Let A and B be $n \times n$ matrices over the complex numbers. Recall that A^* denotes the conjugate of the transpose of A. If $A^*A + B^*B = 0$, prove that $A = B = 0$.

(b) Let A and B be $n \times n$ positive definite Hermitian matrices over the complex numbers. Is it true that the eigenvalues of AB are strictly positive? Prove, or give a counterexample.

Solution: (a) The hypothesis implies that $x^*(A^*A + B^*B)x = 0$ for all x. Consequently

$$x^*A^*Ax = -x^*B^*Bx,$$

but since both sides of the equation are nonnegative, we get $x^*A^*Ax = 0$. This implies $(Ax)^*(Ax) = 0$ for all x, and hence that $A = 0$. Similarly, of course, $B = 0$.

(b) Since B is positive definite, there exists a matrix T such that $B = T^*T$. The eigenvalues of AB are the same as those of

$$T(AB)T^{-1} = T(AT^*T)T^{-1} = TAT^*.$$

If λ is an eigenvalue of TAT^* with eigenvector x then $TAT^*x = \lambda x$, and so

$$x^*TAT^*x = x^*\lambda x.$$

Since A is positive definite we have $x^*TAT^*x > 0$, and it follows that $\lambda > 0$.