Qualifying Exam Problems

1 Let $\mathcal{M}$ be the σ algebra of Lesbegue measu reable substes of the real line. Is it true that for every $E \in \mathcal{M}$, there is an F_σ set A (count able intersection of closed sets) so that $E = A \cup B$ where B is a null set? Prove this or give a counterexample.

IDEA OF SOLUTION: This is true; it uses the regularity of Lebesgue measur e, and the countable additivity. Approximate E on the inside by a sequence of compact sets....

2 Produce an explicit example of a continuous function of two variables $x \geq 1$ and $t \geq 1$, such that

$$\int_1^\infty \left(\int_1^\infty f(x,t)\mathrm{d}x \right) \mathrm{d}t \neq \int_1^\infty \left(\int_1^\infty f(x,t)\mathrm{d}t \right) \mathrm{d}x \ .$$

IDEA OF SOLUTION Violate Fubini's Theorem by having non–integrability. Use something like $f(x,t) = (x-t)e^{-xt}$.

3 Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Suppose that $\{f_n\}$ is a sequence of real valued measurable functions, and that $f(x) = \lim_{n\to\infty} f_n(x)$ for almost every x. Suppose that $\|f\|_1 > 0$. Show that there for some $\epsilon > 0$, there is a strictly positive number b so that for all n *sufficiently large* there is a set E_n with $\mu(E_n) > \epsilon$ and $|f_n(x)| > b$ for all $x \in E_n$.

IDEA OF SOLUTION Usual convergence theorems.

4 Give an example of a dense but not closed linear manifold M in a Banach space X.

5 Prove or find a counter example to the statement: if E is a convex subset of a Hilbert space and $\{x_n\} \subset E$ satisfies $\lim_{n\to\infty} \|x_n\| = \inf\{\|x\| : x \in E\}$ then $\{x_n\}$ is a Cauchy sequence.

6 Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence functions in $L^p(X, \mathcal{S}, \mu)$, $1 < p < \infty$. Suppose that for all $g \in L^q(X, \mathcal{S}, \mu)$ where $q = p/(p-1)$,

$$\lim_{n\to\infty} \int_X f_n g \mathrm{d}\mu = \int_X f g \mathrm{d}\mu$$

for some function $f \in L^p(X, \mathcal{S}, \mu)$. Show that if $\lim_{n\to\infty} \ |f_n\|_p = \|f\|_p$, then there is a subsequence $\{f_{n_k}\}$ so that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ almost everywhere, an d give a counterexample showing that this need not be true without the hypothesis that $\lim_{n\to\infty} \|f_n\|_p = \|f\ |_p$.

7 Let A be an $n \times n$ matrix. Let $\rho(A)$ be the spectra l radius of A, which is, by definition the largest of the absolute values of the eigenvalues of A. Show that $\lim_{n\to\infty} A^n = 0$ if and only if $\rho(A) < 1$. *Do not assume* that A is diagonalizable.

IDEA OF SOLUTION Schur's theorem and "almost diagonalizability".

8 Suppose that $\{\lambda_p\}_{p=1}^\infty$ is a sequence of complex numbers that lie outside the unit disc and that each of $\{\phi_p\}_{p=1}^\infty$ and $\{\theta_p\}_{p=1}^\infty$ is a maximal orthonormal family in a Hilbert space X. Let

$$D(A) = \{x : x \in X \text{and} \ \sum_{p=1}^\infty |\lambda_p|^2 |\langle x, \phi_p\rangle|^2 < \infty\}$$

Define a linear operator (not necesarily bounded) $\mathbf{A}$ by:

$$\mathbf{A}x = \sum_{p=1}^{\infty} \lambda_p \langle x, \phi_p \rangle \theta_p$$

Answer the following questions. If the answer is yes, explain why. If the answer is no, what extra hypothesis must be added to make it yes?

a) Is $\mathbf{A}$ self adjoint?
b) Is there an inverse for $\mathbf{A}$?
c) Is $\mathbf{A}$ a compact operator?
d) Is $\mathbf{A}$ a continuous operator?
e) Is $\mathbf{A}$ a closed operator?
f) Is $\mathbf{A}$ a normal operator?
g) Is the domain of $\mathbf{A}$ all of X?