Qualifying Exam Problems

1. Let $\mathcal{M}$ be the $\sigma$ algebra of Lebesgue measurable subsets of the real line. Is it true that for every $E \in \mathcal{M}$, there is an $F_\sigma$ set $A$ (countable intersection of closed sets) so that $E = A \cup B$ where $B$ is a null set? Prove this or give a counterexample.

**IDEA OF SOLUTION:** This is true; it uses the regularity of Lebesgue measure, and the countable additivity. Approximate $E$ on the inside by a sequence of compact sets....

2. Produce an explicit example of a continuous function of two variables $x \geq 1$ and $t \geq 1$, such that

$$\int_1^\infty \left( \int_1^\infty f(x,t) \, dt \right) \, dx \neq \int_1^\infty \left( \int_1^\infty f(x,t) \, dt \right) \, dx .$$

**IDEA OF SOLUTION** Violate Fubini’s Theorem by having non–integrability. Use something like $f(x,t) = (x-t)e^{-xt}$.

3. Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Suppose that $\{ f_n \}$ is a sequence of real valued measurable functions, and that $f(x) = \lim_{n \to \infty} f_n(x)$ for almost every $x$. Suppose that $\|f\|_1 > 0$. Show that there for some $\epsilon > 0$, there is a strictly positive number $b$ so that for all $n$ sufficiently large there is a set $E_n$ with $\mu(E_n) > \epsilon$ and $|f_n(x)| > b$ for all $x \in E_n$.

**IDEA OF SOLUTION** Usual convergence theorems.

4. Give an example of a dense but not closed linear manifold $M$ in a Banach space $X$.

5. Prove or find a counter example to the statement: if $E$ is a convex subset of a Hilbert space and $\{ x_n \} \subset E$ satisfies $\lim_{n \to \infty} \| x_n \| = \inf \{ \| x \| : x \in E \}$ then $\{ x_n \}$ is a Cauchy sequence.

6. Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Let $\{ f_n \}$ be a sequence functions in $L^p(X, \mathcal{S}, \mu)$, $1 < p < \infty$. Suppose that for all $g \in L^q(X, \mathcal{S}, \mu)$ where $q = p/(p-1)$,

$$\lim_{n \to \infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu$$

for some function $f \in L^p(X, \mathcal{S}, \mu)$. Show that if $\lim_{n \to \infty} \| f_n \|_p = \| f \|_p$, then there is a subsequence $\{ f_{n_k} \}$ so that $\lim_{k \to \infty} f_{n_k}(x) = f(x)$ almost everywhere, and give a counterexample showing that this need not be true without the hypothesis that $\lim_{n \to \infty} \| f_n \|_p = \| f \|_p$.

7. Let $A$ be an $n \times n$ matrix. Let $\rho(A)$ be the spectral radius of $A$, which is, by definition the largest of the absolute values of the eigenvalues of $A$. Show that $\lim_{n \to \infty} A^n = 0$ if and only if $\rho(A) < 1$. Do not assume that $A$ is diagonalizable.

**IDEA OF SOLUTION** Schur’s theorem and “almost diagonalizability”.

8. Suppose that $\{ \lambda_p \}_{p=1}^\infty$ is a sequence of complex numbers that lie outside the unit disc and that each of $\{ \phi_p \}_{p=1}^\infty$ and $\{ \theta_p \}_{p=1}^\infty$ is a maximal orthonormal family in a Hilbert space $X$. Let

$$D(A) = \{ x : x \in X \text{ and } \sum_{p=1}^\infty |\lambda_p|^2 \langle x, \phi_p \rangle^2 < \infty \}$$
Define a linear operator (not necessarily bounded) \( A \) by:

\[
Ax = \sum_{p=1}^{\infty} \lambda_p \langle x, \phi_p \rangle \theta_p
\]

Answer the following questions. If the answer is yes, explain why. If the answer is no, what extra hypothesis must be added to make it yes?

a) Is \( A \) self adjoint?

b) Is there an inverse for \( A \)?

c) Is \( A \) a compact operator?

d) Is \( A \) a continuous operator?

e) Is \( A \) a closed operator?

f) Is \( A \) a normal operator?

g) Is the domain of \( A \) all of \( X \)?