Algebra Comprehensive Exam
Spring 2005

Instructions: Attempt any five questions, and please provide careful and complete answers. If you attempt more questions, specify which five should be graded.

1. Let x and y be elements of a group G such that x has order 3, and y is not the identity and has odd order. If $xyx^{-1} = y^5$, determine the order of y.

   Solution: Using $x^n y x^{-n} = y^{5^n}$ with $n = 3$, we get $y = y^{125}$, so the order of y divides 124. Since the order is an odd integer greater than 1, it must be 31.

2. Let p be an odd prime. If G is a group of order $p(p+1)$ and has more than one p-Sylow subgroup, prove that $p+1$ is a power of 2.

   Solution: The number of p-Sylow subgroups is $s_p \equiv 1 \mod p$ and divides $|G|$, so $s_p = p+1$. Let $P = \langle t \rangle$ be a p-Sylow subgroup. Then $s_p = |G|/|N(P)|$ implies that $|N(P)| = p$, i.e., that $N(P) = P$.

   The p Sylow subgroups contain $(p+1)(p-1)+1 = p^2$ elements, leaving us with p other elements. Since $p+1$ is even, G must contain an element a of order 2. Consider the elements

   $$a,\ tat^{-1},\ t^2at^{-2}, \ldots.\ t^{p-1}at^{-(p-1)}.$$

   If two of these are equal, we can see that a and t commute, which contradicts $N(P) = P$. Hence these are all distinct, so we have p elements of order 2. Since all elements of G have order 1, 2, or p, it follows that $p+1$ must be a power of 2.

3. How many Sylow subgroups are there in a nonabelian group of order 39?

   Solution: The number of p-Sylow subgroups is $s_p \equiv 1 \mod p$ and divides the order of the group, so we must have $s_{13} = 1$. The possibilities for s_3 are 1 and 13, and the nonabelian condition forces $s_3 = 13$.

4. Let $R \subset S$ be commutative integral domains such that every element of S is the root of a monic polynomial with coefficients in R. Prove that R is a field if and only if S is a field.

   Solution: Assume R is a field and take a nonzero element $s \in S$. Then there exist $r_i \in R$ such that

   $$s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n = 0,$$

   where n is least possible. This forces $r_n \neq 0$. But then

   $$s(s^{n-1} + r_1 s^{n-2} + \cdots + r_{n-1}) = -r_n,$$

   and multiplying by $-r_n^{-1}$ shows that s is invertible in S.

   Conversely, assume S is a field and take a nonzero element $r \in R$. Then $r^{-1} \in S$, so there exist $r_i \in R$ such that

   $$r^{-n} + r_1 r^{-n+1} + \cdots + r_{n-1}r^{-1} + r_n = 0.$$

   Multiplying by r^{n-1}, we see that $r^{-1} \in R$.

5. Let x be a transcendental over a field F. If K is a subfield of $F(x)$ properly containing F, prove that x is algebraic over K.

   Solution: Since K is larger than F, it contains a non-constant rational function $f(x)/g(x)$ where $f(x), g(x) \in F[x]$. Since K is a field, there is no loss of generality assuming that $\deg f(x) \geq \deg g(x)$. But then x is algebraic over K since it is a root of the polynomial

   $$f(T) - g(T)\frac{f(x)}{g(x)} \in K[T].$$

6. Let A be a square matrix with integer entries and n be an integer. If each row of A has sum n, prove that n divides the determinant of A.

   Solution: The row-sum condition implies that $(1, \dots, 1)^T$ is an eigenvector of A with eigenvalue n, so the characteristic polynomial has $x - n$ as a factor. Hence n divides the constant term of the characteristic polynomial, which is $\pm \det A$.

7. Determine all 5×5 Hermitian matrices A satisfying $A^5 + 2A^3 + 3A = 6I$, where I denotes the 5×5 identity matrix.

   Solution: Since A is Hermitian, it has real eigenvalues. The only real root of $x^5 + 2x^3 + 3x - 6 = 0$ is 1, so all eigenvalues of A must be 1. Being diagonalizable, A is similar to the identity matrix, and hence must be the identity matrix.

8. If A is a real symmetric matrix, prove that $I + \epsilon A$ is positive definite for sufficiently small real numbers $\epsilon > 0$. (I denotes the identity matrix.)

   Solution: The matrix A is diagonalizable, so let $A = MDM^{-1}$ where D is a diagonal matrix. Then $I + \epsilon A = M(I + \epsilon D)M^{-1}$, so it suffices to show that $I + \epsilon D$ is positive definite for small $\epsilon > 0$. Let $d_1, \dots, d_n$ be the eigenvalues of D. Then $I + \epsilon D$ has positive eigenvalues as long as $1 + \epsilon d_i > 0$ for all i.