Comprehensive Exam, Spring 2005 (Analysis)

<u>**Problem 1:**</u> Show that the closed interval $[0,1]$ is not the disjoint union of a countably infinite family of disjoint nonempty closed sets $\{A_n\}_{n\geq 1}$.

<u>**Solution:**</u> Suppose $[0,1]=\sum_{n\geq 1} A_n$. Denote $U_n = \mathrm{int} A_n$. Then

$$\emptyset \neq X = [0,1]\setminus \sum_{n\geq 1} U_n = \sum_{n\geq 1}(A_n \setminus U_n)$$

is complete with metric inherited from $[0,1]$. Therefore by the Baire Category Theorem there exist n_0 and an open interval V such that $\emptyset \neq X\cap V \subset A_{n_0}$. We claim that then $V\cap U_n = \emptyset$ for $n\neq n_0$.

Let $V=(a,b)$ and let $x\in X\cap V$. Suppose that $y\in V\cap U_n$. Without loss of generality we may assume that $x<y$. Then there must exist a point $z\in A_n\setminus U_n$ such that $a<x<z<y<b$. This implies that $z\in X\cap V$, which implies that $z\in A_{n_0}$. This is a contradiction as A_n and A_{n_0} are disjoint.

Therefore we must have $V\subset X\cup U_{n_0}$ and so $V=(V\cap X)\cup(V\cap U_{n_0})\subset A_{n_0}$. Thus $V\subset U_{n_0}$ which contradicts that $\emptyset\neq X\cap V$.

<u>**Problem 2:**</u> Let $\mu(\Omega)<+\infty$ and let $f_n:\Omega\to\mathbb{R}, n\geq 1$, $f:\Omega\to\mathbb{R}$ be integrable functions such that $0\leq f_n\to f$ a.e., and $\int f_n\,d\mu \to \int f\,d\mu$ as $n\to\infty$. Show that

$$\lim_{a\to\infty}\sup_{n\geq 1}\int_{\{f_n\geq a\}} f_n\,d\mu = 0.$$

<u>**Solution:**</u> Let $\epsilon>0$ and let $\delta>0$ be such that whenever $\mu(E)\leq\delta$ then $\int_E f\,d\mu\leq\epsilon$. (This is possible since f is integrable.) Since $\int f_n\,d\mu \to \int f\,d\mu$, $\int f_n\,d\mu$ are uniformly bounded. Therefore there exists a_1 such that $\mu(\{f_n\geq a_1\})\leq\delta$ for all $n\geq 1$.

Denote $B_n=\{|f_n-f|\leq\epsilon\}$. By Egoroff's Theorem there exists n_1 such that $\mu(\Omega\setminus B_n)\leq\delta$ for $n\geq n_1$. Finally let n_2 be such that

$$\int_\Omega (f_n-f)\,d\mu\leq\epsilon$$

for $n\geq n_2$. Then for $n\geq\max(n_1,n_2)$ and $a\geq a_1$ we have

$$
\begin{aligned}
&\int_{\{f_n\ge a\}} f_n\,d\mu \le \int_{\{f_n\ge a\}\cap B_n} f_n\,d\mu + \int_{\Omega\setminus B_n} f_n\,d\mu \\
&\le \int_{\{f_n\ge a\}\cap B_n} (|f_n - f| + f)\,d\mu + \int_{\Omega\setminus B_n} (f_n - f)\,d\mu + \int_{\Omega\setminus B_n} f\,d\mu \\
&\le \int_{\{f_n\ge a\}\cap B_n} (\epsilon + f)\,d\mu + \int_{\Omega} (f_n - f)\,d\mu + \int_{B_n} |f_n - f|\,d\mu + \int_{\Omega\setminus B_n} f\,d\mu \\
&\quad \le \delta\epsilon + \epsilon + \epsilon + \mu(\Omega)\epsilon + \epsilon = \epsilon(3 + \delta + \mu(\Omega)).
\end{aligned}
$$

We now take a_2 such that

$$\int_{\{f_n\ge a_2\}} f_n\,d\mu \le \epsilon \quad \text{for } n = 1, ..., \max(n_1, n_2).$$

Then for $a \ge \max(a_1, a_2)$ and all $n \ge 1$ we have

$$\sup_{n\ge 1} \int_{\{f_n\ge a\}} f_n\,d\mu \le \epsilon(3 + \delta + \mu(\Omega))$$

which proves the claim.

Problem 3: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $0 < p < 1$ and $-\infty < q < 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be positive measurable functions such that f^p and g^q are integrable. Assume also that fg is integrable. Prove that

$$\left(\int_\Omega f^p\,d\mu\right)^{\frac{1}{p}} \left(\int_\Omega g^q\,d\mu\right)^{\frac{1}{q}} \le \int_\Omega fg\,d\mu.$$

Solution: Set $r = \frac{1}{p}, s = -\frac{q}{p}$. Then

$$\frac{1}{r} + \frac{1}{s} = p - \frac{p}{q} = p(1 - \frac{1}{q}) = 1.$$

Moreover $(fg)^p \in L^r, g^{-p} \in L^s$. Therefore by Hölder inequality

$$
\begin{aligned}
\int_\Omega f^p\,d\mu = \int_\Omega (fg)^p g^{-p}\,d\mu &\le \left(\int_\Omega (fg)^{\frac{p}{p}}\,d\mu\right)^p \left(\int_\Omega g^{-p(-\frac{q}{p})}\,d\mu\right)^{-\frac{p}{q}} \\
&= \left(\int_\Omega fg\,d\mu\right)^p \left(\int_\Omega g^q\,d\mu\right)^{-\frac{p}{q}}
\end{aligned}
$$

Therefore we obtain that

$$\left(\int_\Omega f^p\,d\mu\right)^{\frac{1}{p}} \le \left(\int_\Omega fg\,d\mu\right) \left(\int_\Omega g^q\,d\mu\right)^{-\frac{1}{q}}$$

which proves the inequality.

Problem 4: Let $\{x_n\}$ be a sequence of pairwise orthogonal vectors in a Hilbert space H. Show that the following are equivalent:
(a) $\sum_{n=1}^{\infty} x_n$ converges in the norm topology of H.
(b) $\sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$.
(c) $\sum_{n=1}^{\infty} < x_n, y >$ converges for every $y \in H$.

Solution: $(b) \Rightarrow (a)$: Since $< x_i, x_j >= 0$ if $i \neq j$, we have

$$\|x_n + ... + x_m\|^2 = \|x_n\|^2 + ... + \|x_m\|^2$$

whenever $n \leq m$. Therefore (b) implies that the partial sums of $\sum x_n$ form a Cauchy sequence in H which, by completeness of H implies (a).
$(a) \Rightarrow (c)$ By the Schwarz inequality

$$| < x_n, y > +...+ < x_m, y > | \leq \|x_n + ... + x_m\| \|y\|$$

whenever $n \leq m$ and so the series in (c) converges.
$(c) \Rightarrow (b)$ Denote $y_n = x_1 + ... + x_n$ for $n \geq 1$. Then (c) implies that (y_n) converges weakly in H and so there exists C such that $\|y_n\| \leq C$. But since the x_n are pairwise orthogonal we have

$$\|x_1\|^2 + ... + \|x_n\|^2 = \|x_1 + ... + x_n\|^2 = \|y_n\|^2 \leq C^2$$

for every $n \geq 1$. This implies (b).

Problem 5: Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}^n$ such that $m(E_k) \to 0$, where $m(A)$ denote the Lebesgue measure of $A \subset \mathbb{R}^n$.

(a) Show that there exists a subsequence $\{E_{k_n}\}$ such that $\limsup_{n\to\infty} E_{k_n} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_{k_n}$ has Lebesgue measure zero.

(b) In general, does $m(E_k) \to 0$ imply $m(\limsup_{k\to\infty} E_k) = 0$?

Solution:

(a) Choose an increasing sequence $k_n \uparrow \infty$ such that $m(E_{k_n}) < 2^{-n}$. In order for this to hold, we just need to pick large enough k_n, since $\lim_{k\to\infty} m(E_k) = 0$.

Denote $E = \limsup_{n\to\infty} E_{k_n}$. For every N, we have $E \subset \bigcup_{n=N}^{\infty} E_{k_n}$ and hence

$$m(E) \leq m(\bigcup_{n=N}^{\infty} E_{k_n}) \leq \sum_{n=N}^{\infty} m(E_{k_n}).$$

From $m(E_{k_n}) < 2^{-n}$ it follows that $\sum_{n=1}^{\infty} m(E_{k_n}) < \sum_{n=1}^{\infty} 2^{-n} < \infty$. Hence, the partial sum $\sum_{n=N}^{\infty} m(E_{k_n})$ converges to 0 as $N \to \infty$. Thus, $m(E) = 0$.

(b)

(b) Answer: No.

Example: Let

$$\begin{aligned} & E_1 = [0, 1/2], E_2 = [1/2, 1], \\ & E_3 = [0, 1/3], E_4 = [1/3, 2/3], E_5 = [2/3, 1], \\ & \cdots \\ & [0, 1/n], [1/n, 2/n], \cdots, [(n-1)/n, 1], \\ & \cdots . \end{aligned}$$

Clearly, $m(E_k) \to 0$ as $k \to \infty$. However, every point $x \in [0, 1]$ belongs to infinitely many E_k's; that is, $\limsup_{k\to\infty} E_k = [0, 1]$.

Problem 6: Let (X, d) be a compact metric space and $f : X \to X$ be a map satisfying

$$d(f(x), f(y)) < d(x, y) \qquad \text{for all } x \in X, y \in X, x \neq y. \tag{$*$}$$

(a) Show that the function $g : X \to \mathbb{R}$, $g(x) = d(x, f(x))$, is uniformly continuous on X.

(b) Show that $\min\limits_{x \in X} g(x) = 0$.

(c) Show that there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.

Solution:

(a) Let $x \in X, y \in X$. Estimate using the triangle inequality and $(*)$:

$$\begin{aligned} |d(x, f(x)) - d(y, f(y))| &\leq |d(x, f(x)) - d(y, f(x))| + |d(y, f(x)) - d(y, f(y))| \\ &\leq d(x, y) + d(f(x), f(y)) \\ &\leq 2d(x, y). \end{aligned}$$

Thus, g is Lipschitz continuous with Lipschitz constant 2. This shows the uniform continuity of g.

(b) Since g is continuous on a compact space X, it attains its minimum value on X. Let $x_0 \in X$ be a minimum point. Suppose on the contrary that $g(x_0) = d(x_0, f(x_0)) > 0$. This means $f(x_0) \neq x_0$. Condition $(*)$ implies

$$d(f(x_0), f^2(x_0)) < d(x_0, f(x_0)).$$

That is $g(f(x_0)) < g(x_0)$, contradicting $g(x_0) = \min g$.

(c) Existence: Part (b) shows that at a minimum point x_0 of g, we have $f(x_0) = x_0$.

Uniqueness: Let $x \neq x_0$. Condition $(*)$ implies $d(f(x), f(x_0)) < d(x, x_0)$. Since $f(x_0) = x_0$, it follows that $d(f(x), x_0) < d(x, x_0)$. Thus, $f(x) \neq x$.

Problem 7: Let $f \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. For $y \in \mathbb{R}$, define

$$u(y) = \int_{\mathbb{R}} |f(x, y)| dx.$$

For $x \in \mathbb{R}$, define

$$v(x) = \left(\int_{\mathbb{R}} |f(x, y)|^2 dy \right)^{1/2}.$$

(a) Show that $u \in L^1(\mathbb{R})$ and $v \in L^2(\mathbb{R})$.

(b) Show that for any Lebesgue measurable function $g \geq 0$,

$$\int_{\mathbb{R}} u(y) g(y) dy \leq \left(\int_{\mathbb{R}} v(x) dx \right) \left(\int_{\mathbb{R}} g(y)^2 dy \right)^{1/2}.$$

(c) Show that

$$\int_{\mathbb{R}} u(y)^2 dy \leq \int_{\mathbb{R}} v(x) dx.$$

(In particular, if $\int_{\mathbb{R}} u(y)^2 dy = \infty$, then $\int_{\mathbb{R}} v(x) dx = \infty$.)

Solution:

(a) For u: Since

$$\|f\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |f| < \infty,$$

the Fubini-Tonelli theorem implies that the function

$$u(y) = \int_{\mathbb{R}} |f(x, y)| dx$$

is Lebesgue integrable on $\mathbb{R}$ with $\|u\|_{L^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R}^2)}$.

For v: Since

$$\|f\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |f|^2 < \infty,$$

the Fubini-Tonelli theorem implies that the function

$$v(x)^2 = \int_{\mathbb{R}} |f(x, y)|^2 dy$$

is Lebesgue integrable on $\mathbb{R}$ with $\int_{\mathbb{R}} v(x)^2 dx = \int_{\mathbb{R}^2} |f|^2$. This means $v \in L^2(\mathbb{R})$.

(b) By Fubini-Tonelli and Cauchy-Schwarz,

$$\begin{aligned}
\int_{\mathbb{R}} u(y)g(y)dy &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|g(y)dx \right) dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|g(y)dy \right) dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|^2 dy \right)^{1/2} \left(\int_{\mathbb{R}} g(y)^2 dy \right)^{1/2} dx \\
&= \left(\int_{\mathbb{R}} v(x)dx \right) \left(\int_{\mathbb{R}} g(y)^2 dy \right)^{1/2}.
\end{aligned}$$

(c) If $u \in L^2(\mathbb{R})$, just let $g = u$ in (b) and we are done.

If $u \notin L^2(\mathbb{R})$, we have $\int_{\mathbb{R}} u(y)^2 dy = \infty$. We need to show that $\int_{\mathbb{R}} v(x)dx = \infty$. Let

$$u_n(x) = \begin{cases} u(x) & \text{for } |y| < n \text{ where } u(y) < n; \\ n & \text{for } |y| < n \text{ where } n \leq u(y) \leq \infty; \\ 0 & \text{for } |y| \geq n. \end{cases}$$

Then, every $u_n \in L^2(\mathbb{R})$. Moreover, $u_n(y) \uparrow u(y)$ pointwise everywhere in $\mathbb{R}$. The monotone convergence theorem implies

$$\int_{\mathbb{R}} u_n(x)^2 dx \uparrow \int_{\mathbb{R}} u(x)^2 dx = \infty.$$

Applying (b) to $g = u_n$, we obtain

$$\int_{\mathbb{R}} u(y)u_n(y)dy \leq \left(\int_{\mathbb{R}} v(x)dx \right) \left(\int_{\mathbb{R}} u_n(x)^2 dx \right)^{1/2}.$$

Since $u_n \leq u$, the integral on the left hand side is no less than $\int_{\mathbb{R}} u_n(x)^2 dx$. Hence,

$$\left(\int_{\mathbb{R}} u_n(x)^2 dx \right)^{1/2} \leq \int_{\mathbb{R}} v(x)dx.$$

Passing to the limit $n \to \infty$, this shows $\int_{\mathbb{R}} v(x)dx = \infty$.

Problem 8: Let (Ω, Σ, μ) be a measure space with $\mu(\Omega) < \infty$. Let $\psi : \Omega \to \Omega$ be a map satisfying

(i) $\psi^{-1}(A) \in \Sigma$ for any $A \in \Sigma$;

(ii) if $A \in \Sigma$ and $\mu(A) = 0$, then $\mu(\psi^{-1}(A)) = 0$.

Prove the following statements (a)-(b):

(a) There exists a unique nonnegtaive $h \in L^1(\Omega, \mu)$ such that

$$\mu(\psi^{-1}(A)) = \int_A h(x)d\mu,$$

for any $A \in \Sigma$.

(b) For any $f \in L^1(\Omega, \mu)$.

$$\int_\Omega f \circ \psi(x)d\mu = \int_\Omega f(x)h(x)d\mu.$$

Solution:

(a) Consider $\nu(A) = \mu(\psi^{-1}(A))$ for $A \in \Sigma$. ν defines a measure on Σ: if $\cup_n A_n$ is a countable disjoint union of sets $A_n \in \Sigma$, then so is $\psi^{-1}(\cup_n A_n) = \cup_n \psi^{-1}(A_n)$. Hence,

$$\nu(\cup_n A_n) = \mu(\cup_n \psi^{-1}(A_n)) = \sum_n \mu(\psi^{-1}(A_n)) = \sum_n \nu(A_n).$$

Moreover, condition (ii) implies the absolutely continuity of ν with respect to μ.

By the Radon-Nikodym theorem, there exists a unique nonnegtaive $h \in L^1(\Omega, \mu)$ such that

$$\nu(A) = \int_A h(x)d\mu,$$

for any $A \in \Sigma$.

(b) It suffices to prove the integral identity for a real-valued nonnegative L^1 function f. (For a complex-valued f, consider $f = \operatorname{Re} f + i \operatorname{Im} f$. For a real-valued sign-changing f, consider $f = f_+ - f_-$.)

First for $f \in L^1(\Omega)$ that are nonnegative and simple, the required integral identity is an immediate consequence of part (a). Details follow: Let f be a finite sum: $f = \sum_n a_n \chi_{A_n}$, where A_n's are measurable and pairwise disjoint and a_n are distinct values.

$$\begin{aligned}
\int_\Omega f \circ \psi(x)d\mu &= \int_\Omega \sum_n a_n \chi_{A_n} \circ \psi(x)d\mu \\
&= \sum_n a_n \int_\Omega \chi_{\psi^{-1}(A_n)}(x)d\mu \\
&= \sum_n a_n \mu(\psi^{-1}(A_n)) \\
&= \sum_n a_n \int_{A_n} h(x)d\mu \qquad \text{(used (a) here)} \\
&= \sum_n a_n \int_\Omega \chi_{A_n}(x)h(x)d\mu \\
&= \int_\Omega f(x)h(x)d\mu.
\end{aligned}$$

Next let $f \in L^1(\Omega)$ be nonnegative. Take a sequence of nonnegative simple functions $f_n \in L^1(\Omega)$ such that $f_n(x) \uparrow f(x)$ a.e. in Ω. For f_n we have already shown that

$$\int_\Omega f_n \circ \psi(x) d\mu = \int_\Omega f_n(x) h(x) d\mu. \tag{$*$}$$

Now we take the limit as $n \to \infty$. Since $f_n(x) \uparrow f(x)$ a.e. in Ω and h is nonnegative, we have $f_n(x)h(x) \uparrow f(x)h(x)$ a.e. in Ω. By the Monotone Convergence Theorem, the right hand side of $(*)$ converges to $\int_\Omega f(x)h(x)d\mu$. Similarly, if we can show

$$f_n \circ \psi(x) \uparrow f \circ \psi(x) \qquad \text{a.e. in } \Omega, \tag{$**$}$$

then the left hand side of $(*)$ converges to $\int_\Omega f \circ \psi(x) d\mu$ and we are done. Now we prove $(**)$. By the assumption, there is a set $Z \in \Sigma$ such that $\mu(Z) = 0$ and $f_n(x) \uparrow f(x)$ for every $x \in \Omega \setminus Z$. Hence, $f_n \circ \psi(x) \uparrow f \circ \psi(x)$ for every x such that $\psi(x) \in \Omega \setminus Z$, or , equivalently, for every $x \in \psi^{-1}(\Omega \setminus Z) = \Omega \setminus \psi^{-1}(Z)$. Condition (ii) guarantees $\mu(\psi^{-1}(Z)) = 0$. Thus, $(**)$ holds and the proof is complete.