Analysis Comprehensive Exam
January 13, 2006

Complete FIVE of the SEVEN problems below.

1. Let X be a compact metric space, and let $\{x_n\}$ be a sequence in X. Suppose that every convergent subsequence of $\{x_n\}$ converges to the same element x_0 of X. Show that $\{x_n\}$ converges to x_0.

Solution:

We claim that $\{x_n\}$ converges to x_0. Suppose not. Then there exists an open neighborhood O of x_0 such that $x_n \notin O$ for infinitely many n. It follows that there exists a subsequence s of $\{x_n\}$ such that every term of s lies in $X - O$. Since O is open, $X - O$ is closed in X. Since X is compact, s has a subsequence t that converges, and its limit must lie in the closed set $X - O$. In particular, the limit of t is not x_0. But t is also a subsequence of $\{x_n\}$, which contradicts the fact that every convergent subsequence of $\{x_n\}$ converges to x_0.

2. Let $\{f_n\}$ be a sequence of continuous non–negative functions that converges pointwise on $[0,1]$ to a function f.
   (a) Suppose that $\int_0^1 f\,dx = 0$ and the sequence of integrals $\int_0^1 f_n\,dx$ is bounded. Must we have $\lim_{n\to\infty} \int_0^1 f_n\,dx = 0$? Give a proof or a counterexample.
   (b) Suppose that $\int_0^1 f\,dx = 0$ and that $\{f_n\}$ is non-increasing. Must we have $\lim_{n\to\infty} \int_0^1 f_n\,dx = 0$? Give a proof or a counterexample.
   (c) Suppose that f is equal to zero everywhere, and that $\{f_n\}$ is non-increasing. Show that $\{f_n\}$ is uniformly convergent on [0,1].

Solution:

(a) No. Let f_1 take the values $f_1(0) = 0$, $f_1(\frac{1}{2}) = 2$, $f_1(1) = 0$, and be piecewise linear in between. Then f_1 is non negative, and has integral one. For $n = 2, 3, \ldots$ set

$$f_n(x) = \begin{cases} 0 & \frac{1}{n} < x \le 1 \\ n f_1(nx) & 0 \le x \le \frac{1}{n} \end{cases}$$

Then, the integral of each f_n is one. All functions are continuous, and they converge pointwise to zero.

(b) Yes. The functions $\{f_n\}$ are bounded above by f_1 and below by zero, so this follows from the monotone convergence theorem (or the dominated convergence theorem, or the bounded convergence theorem...).

(c) Fix $\epsilon > 0$. For each $x \in [0,1]$ there exists a positive integer n_x such that $f_{n_x} < \epsilon$ whenever $n \geq n_x$. For each x, the set $U_x = \{f_{n_x} < \epsilon\}$ is an open neighborhood of x in $[0,1]$ (with the relative topology), since each f_n is continuous. Since $\{f_n\}$ is monotone, $z \in U_x$ implies that $f_n(z) < \epsilon$ for all n with $n \geq n_x$. Since the collection $\{U_x : x \in [0,1]\}$ covers the compact space $[0,1]$, there is a finite subcover $\{U_{x_1}, \ldots, U_{x_N}\}$.

Let $M = \sup\{N_{x_1}, \ldots, N_{x_N}\}$. Let $n \geq M$ and $y \in [0,1]$. Then $y \in U_{x_j}$ for some $1 \leq j \leq N$. Hence $f_n(y) < \epsilon$. Since y was arbitrary in $[0,1]$, we have $f_n(y) < \epsilon$ for all y in $[0,1]$ and all $n \geq M$. It follows that $\{f_n\}$ converges uniformly on $[0,1]$ to zero.

3. Let $g(x) = (x \log x)^{-1}$ on the interval $[3, \infty)$. Let $f_n = c_n \chi_{A_n}$ where $c_n \geq 0$ and A_n is a measurable subset of $[3,\infty)$. Assume that $0 \leq f_n \leq g(x)$, and that $f_n \longrightarrow 0$ a. e.

   (a) Show that for all $3 < N < \infty$, we have $\int_3^N f_n(x)\, dx \longrightarrow 0$.

   (b) Show that $\int_3^\infty f_n(x)\, dx \longrightarrow 0$.

Solution:

(a) While $g(x)$ is not integrable on $[3,\infty)$, it is integrable on every finite length interval $[3, N]$. Lebesgue Dominated Convergence Theorem then implies that $\int_3^N f_n(x)\, dx \longrightarrow 0$.

(b) Fix a large fixed $N > 3$. Since $0 \leq f_n \leq g$, observe that we must have $0 < c_n < [(3 + |A_n|)\log(3+|A_n|)]^{-1}$. Moreover, if f_n is non zero on the interval $[N, \infty)$, we must have $c_n < [(3+|A_n|)\log N]^{-1}$ Therefore,

$$\int_N^\infty f_n\, dx \leq \frac{|A_n|}{(3+|A_n|)\log N} \leq (\log N)^{-1}.$$

But, by the first part, we have $\int_3^N f_n(x)\, dx \longrightarrow 0$, hence

$$\limsup_n \int_3^\infty f_n(x)\, dx \leq (\log N)^{-1}.$$

As $N > 3$ was arbitrary, we have finished the proof.

4. For each set S let $\mathcal{P}(S)$ denote the power set of S, i.e., the set of all subsets of S.

   (a) Show that if S is infinite, then $\mathcal{P}(S)$ is uncountable.

   (b) Let $\mathcal{C}$ be the product of countably many copies of the two-point space $\{0,1\}$. Show that $\mathcal{C}$ is uncountable.

(c) Show that the Cantor ternary set is uncountable.

The Cantor ternary set is $\bigcap_{n=0}^{\infty} C_n$ where C_n is the sequence of closed sets

$$C_0 = [0,1]\,, \quad C_1 = C_0 - (\tfrac{1}{3}, \tfrac{2}{3})\,, \quad C_2 = C_1 - \{(\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})\}\,, \ldots$$

Solution:

(a) Every infinite set S contains a countably infinite set T, for which we have $\mathcal{P}(T) \subset \mathcal{P}(S)$. If $\mathcal{P}(T)$ is uncountable, then $\mathcal{P}(S)$ is uncountable, so it suffices to prove the result for countably infinite S. Suppose then that S is countably infinite, and let $s_1, s_2, \ldots$ be an enumeration of S. Since each singleton set $\{s_i\}$ is an element of $\mathcal{P}(S)$, $\mathcal{P}(S)$ is clearly infinite. Let $f(i) = S_i$ be any injection of the positive integers into $\mathcal{P}(S)$. We form a subset W of S as follows: for each positive integer i, we let s_i be an element of W if and only if s_i is not an element of S_i. Then for each i, W is distinct from S_i, since s_i lies in W if and only if s_i does not lie in s_i. This means that f does not map onto $\mathcal{P}(S)$. Thus $\mathcal{P}(S)$ is not in one-to-one correspondence with the positive integers. Since $\mathcal{P}(S)$ is infinite, it is uncountable.

(b) Observe that $\mathcal{C}$ is in one-to-one correspondence with the power set $\mathcal{P}(\mathbb{N})$ of the positive integers: if x is an element of $\mathcal{C}$, then $S(x) = \{i : x_i = 1\}$ is a subset of N, and $x \to S(x)$ is a bijection of $\mathcal{C}$ with $\mathcal{P}(\mathbb{N})$. The result now follows from part 1.

(c) The Cantor ternary set is the set of real numbers between zero and one whose ternary expansion uses no one. Thus it is in one-to-one correspondence with the product of countably infinitely many copies of the two-point space $\{0, 2\}$, which in turn is in one-to-one correspondence with the product $\mathcal{C}$ of part 2.

5. A function is *absolutely continuous* on $[0,1]$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that for all disjoint subintervals $[a_j, b_j) \subset [0,1]$, $j = 1, 2, \ldots$, with $\sum_j (b_j - a_j) < \delta$, we have $\sum_j |f(b_j) - f(a_j)| < \epsilon$.

   (a) Show that every absolutely continuous function is of bounded variation.

   (b) Give an example of a continuous function which is not absolutely continuous.

Solution:

This is a very standard exercise.

6. (a) Give an example of a measure space (X, Ω, μ) for which we have the inclusion $L^p(X) \subseteq L^q(X)$ for all $1 \le p < q \le \infty$.

(b) Give an example of a measure space (X, Ω, μ) for which we have the inclusion $L^p(X) \supseteq L^q(X)$ for all $1 \le p < q \le \infty$.

In both instances, the space $L^p(X)$ should be infinite dimensional. And the inclusions are to be proved.

Solution:

(a) Take X to be the integers $\mathbb{N}$, and the measure space is the one associated to counting measure on $\mathbb{N}$. That is, $L^p(X) = \ell^p(\mathbb{N})$. For $1 \le p < q \le \infty$, and $f \in \ell^p(\mathbb{N})$, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} |f(n)|^q &\le \sup_{n\in\mathbb{N}} |f(n)|^{q-p} \times \sum_{n\in\mathbb{N}} |f(n)|^p \\ &\le \Big(\sum_{n\in N} |f(n)|^p\Big)^{q/p-1} \|f\|_p^p \\ &\le \|f\|_p^{q/p+p-1}.\end{aligned}$$

That is, $f \in \ell^p$ implies $f \in \ell^q$.

(b) Take $X = [0,1]$, Ω to be the Lebesgue measurable sets of $[0,1]$ and μ to be Lebesgue measure. For $1 \le p < q \le \infty$, and $f \in \ell^p(\mathbb{N})$, we have

$$\begin{aligned}\|f\|_p^p &= \int_0^1 |f|^p \, dx \\ &= \int_{|f|\le 1} |f|^p \, dx + \int_{|f|\ge 1} |f|^p \, dx \\ &\le 1 + \int_{|f|\ge 1} |f|^q \, dx \\ &\le 1 + \|f\|_q^q.\end{aligned}$$

That is, $f \in L^q([0,1])$ implies $f \in L^p([0,1])$.

7. Consider $[0,1]$ with addition modulo one. Show that a function $f : [0,1) \longrightarrow \mathbb{C}$, with $f \in L^4(0,1)$ for which

$$\int_0^1 \Big| \int_0^1 f(x)\overline{f(x-s)} \, dx \Big|^2 \, ds = 0$$

is zero a. e. [Hint: Use the exponential basis on $L^2(0,1)$.]

Solution:

Write the function $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \mathrm{e}^{2\pi inx}$, where $\widehat{f}(n) = \int_0^1 f(x)\, \mathrm{e}^{-2\pi inx}\ dx$. Observe that $f(x)\overline{f(x-s)}$ is square integrable, and that

$$\begin{aligned}\int_0^1 f(x)\overline{f(x-s)} &= \sum_{m,n\in\mathbb{Z}} \widehat{f}(m)\overline{\widehat{f}(n)}\, \mathrm{e}^{2\pi ins} \int_0^1 \mathrm{e}^{2\pi i(m-n)x}\, dx \\ &= \sum_n |\widehat{f}(n)|^2\, \mathrm{e}^{2\pi ins}\ .\end{aligned}$$

The assumption is that this last sum has zero L^2 norm. That means that each Fourier coefficient must be zero. Hence f is zero a. e.