Solutions to Analysis Comprehensive Exam - January 2007

A linear map T on a normed space X is called a *strict contraction* in case there is constant $c < 1$ so that $\|Tf\| \le c\|f\|$ for all $f \in X$.

For $1 \le p \le \infty$, define $T : L^p([0,1]) \to L^p([0,1])$ by

$$Tf(t) = \int_0^t f(s)\mathrm{d}s \ .$$

(a) Show that for $1 < p < \infty$, T is a strict contraction on $L^p([0,1])$.

(b) Show for $p = 1$ and $p = \infty$, T^2 is a strict contraction on $L^p([0,1])$, but T is not.

(c) Show that for all $f \in L^1([0,1])$, the sequence $\{T^n f\}_{n\ge1}$ converges in $L^1([0,1])$.

Solution: (a) By Hölder's inequality, for all t,

$$|Tf(t)| = \int_{[0,1]} 1_{[0,t]}(s)f(s)\mathrm{d}s \le t^{1/p'}\|f\|_p \ .$$

Hence

$$\|Tf\|_p \le \left(\int_0^1 t^{p/p'}\mathrm{d}t\right)^{1/p} \|f\|_p = (1/p)^{1/p}\|f\|_p \ .$$

Hence we have $c_p = (1/p)^{1/p}$.

(b) For $p = 1$, we have that

$$|Tf(t)| \le \int_0^1 |f(s)|\mathrm{d}s = \|f\|_1 \ .$$

To see that this bound is sharp, consider $f_n(s) = n$ on $[0,1/n]$ with $f_n(s) = 0$ for $s > 1/n$. Then $Tf_n(t) = \|f\|_1$ for all $t \ge 1/n$. Thus, $\|Tf_n\|_1 \ge (1-1/n)\|f_n\|_1$, and so T is not a strict contraction on L^1.

However, using the bound $|Tf(t)| \le \|f\|_1$, we have that

$$T^2 f(t) \le t\|f\|_1 \ ,$$

and so

$$\|T^2 f\|_1 \le (1/2)\|f\|_1 \ .$$

Hence T^2 is a strict contraction on L^1.

For $p = \infty$, note that

$$|Tf(t)| \le \int_0^t |f(s)|\mathrm{d}s \le t\|f\|_\infty \ . \qquad (*)$$

There is equality if $f(t) = 1$ for all t, and in this case we have $\|Tf\|_\infty = \|f\|_\infty$, so that T is not a strict contraction on L^∞. However, from $(*)$,

$$T^2 f(t) \le (t^2/2)\|f\|_\infty \ ,$$

and so

$$\|T^2 f\|_\infty \le (1/2)\|f\|_\infty \ .$$

(c) It follows from the above that

$$\|T^{2k+1}f\|_1 \leq \|T^{2k}f\|_1 \leq (1/2)^k\|f\|_1 \ .$$

Hence $T_n f$ converges to zero in L^1.

Question 2.
For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be integrable. Assume that

$$\lim_{k\to\infty} f_k = f \text{ a.e. in } [0,1],$$

where f is integrable over $[0,1]$. Prove that the following are equivalent:
(a)

$$\lim_{k\to\infty} \int_0^1 |f_k - f| = 0.$$

(b) $\{f_k\}$ are uniformly integrable, that is the following property is true: $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$E \subset [0,1] \text{ and } |E| < \delta \Rightarrow \left| \int_E f_k \right| < \varepsilon \text{ for all } k \geq 1. \tag{1}$$

(You may assume that integrable functions have absolutely continuous integrals. That is, if g is integrable over $[0,1]$, then $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$E \subset [0,1] \text{ and } |E| < \delta \Rightarrow \left| \int_E g \right| < \varepsilon.)$$

Solution
(b)$\Rightarrow$(a)
Let $\varepsilon > 0$. Choose $\delta > 0$ as in (b). By Egorov's theorem, there exists a (closed) set $F \subset [0,1]$ such that

$$E = [0,1] \setminus F$$

has $|E| < \delta$ and $\{f_k\}$ converges uniformly to f on F. Then as $k \to \infty$,

$$\int_F |f - f_k| \leq |F| \sup_F |f - f_k| \to 0. \tag{2}$$

We now bound the integral over the complementary range $E = [0,1] \setminus F$. We use

$$\begin{aligned} &\int_E |f - f_k| \\ &\leq \int_E |f| + \int_E |f_k| \\ &= \int_E |f| + \int_{E_k^+} f_k + \int_{E_k^-} (-f_k), \end{aligned} \tag{3}$$

where

$$\begin{aligned} E_k^+ &= \{x \in E : f_k(x) \geq 0\}; \\ E_k^- &= \{x \in E : f_k(x) < 0\}. \end{aligned}$$

Both these sets are measurable, as f_k and E are. Then

$$\left| E_k^\pm \right| \leq |E| < \delta,$$

so our hypothesis gives

$$\left|\int_{E_k^+} f_k\right| + \left|\int_{E_k^-} f_k\right| < 2\varepsilon.$$

Combining this, (2), and (3), we obtain

$$\begin{aligned}
& \limsup_{k\to\infty} \int_0^1 |f - f_k| \\
\leq \; & \limsup_{k\to\infty} \left(\int_F |f - f_k| + \int_E |f - f_k| \right) \\
\leq \; & \int_E |f| + 2\varepsilon.
\end{aligned}$$

Finally, as f is integrable, its integral is absolutely continuous, so

$$\int_E |f| \to 0 \text{ as } |E| \to 0+.$$

(Alternatively, one can use dominated convergence). We deduce (1).
(a)$\Rightarrow$(b)
Let $\varepsilon > 0$. There exists N such that

$$k > N \Rightarrow \int_0^1 |f_k - f| < \varepsilon/2.$$

Since f and $f_1, f_2, ..., f_N$ are integrable, their integrals are absolutely continuous, so there exists $\delta > 0$ such that

$$|E| < \delta \Rightarrow \int_E |f| < \frac{\varepsilon}{2} \text{ and } \int_E |f_k| < \frac{\varepsilon}{2}, k \leq N.$$

Then if $|E| < \delta$ and $k > N$,

$$\begin{aligned}
\left|\int_E f_k\right| & \leq \int_E |f_k| \\
& \leq \int_E |f_k - f| + \int_E |f| \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

If $k \leq N$, we already have what we need. ■

Question 3. Let A be a subset of a metric space X. Suppose that every continuous function on A is uniformly continuous. Show that A is closed. Is A necessarily compact?.
Solution: For any y in the complement of A, define f on A by

$$f(x) = \frac{1}{d(x,y)} \ .$$

Then f is the composition of continuous functions, and hence is continuous.

Suppose that y is a limit point of A that is in the complement of A. Then for any $\epsilon > 0$, we can find an x_1 in A with $d(x_1, y) < \epsilon$. We can then find an x_2 in A with $d(x_2, y) < d(x_1, y)/2$. Then

$$f(x_2) - f(x_1) = \frac{1}{d(x_2,y)} - \frac{1}{d(x_1,y)} \geq \frac{1}{d(x_1,y)} \geq \frac{1}{\epsilon} \ .$$

However,

$$d(x_1, x_2) \leq d(x_1, y) + d(x_2, y) \leq 2\epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, this is incompatible with the unifom continuity of f. Hence there are no limit points of A in the complement of A, and so A is closed.

A need not be compact. consider the set of natural numbers with the metric inherited from the reals.

Question 4.
Let $a > 1$. Define a sequence $\{x_n\}$ by $x_0 = 1$ and

$$x_{n+1} = \frac{1}{a + x_n}, n \geq 0.$$

Prove that

$$\lim_{n \to \infty} x_n$$

exists and evaluate the limit. **For example**, you may use the following steps:
(a) Show that for $n \geq 1$,

$$x_{n+1} - x_n = -\frac{x_n - x_{n-1}}{(a + x_n)(a + x_{n-1})}.$$

(b) Prove that

$$|x_{n+1} - x_n| \leq \frac{|x_n - x_{n-1}|}{a^2}.$$

(c) Prove that

$$\sum_{n=0}^{\infty} |x_{n+1} - x_n| < \infty.$$

(d) Hence complete the problem.
Solution
(a)

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{a + x_n} - \frac{1}{a + x_{n-1}} \\ &= \frac{x_{n-1} - x_n}{(a + x_n)(a + x_{n-1})}. \end{aligned}$$

(b) Note that as $a > 0$, and as $x_0 = 1 > 0$, so all $x_n > 0$, by induction. Then from (a),

$$\begin{aligned} |x_{n+1} - x_n| &= \frac{|x_{n-1} - x_n|}{(a + x_n)(a + x_{n-1})} \\ &\leq \frac{|x_{n-1} - x_n|}{a^2}, \end{aligned}$$

so the desired inequality follows.
(c) We iterate the inequality of (b):

$$\begin{aligned} |x_{n+1} - x_n| &\leq \frac{|x_n - x_{n-1}|}{a^2} \\ &\leq \left(\frac{1}{a^2}\right)^2 |x_{n-1} - x_{n-2}| \\ &\leq \ldots \\ &\leq \left(\frac{1}{a^2}\right)^n |x_1 - x_0|. \end{aligned}$$

Then as $a > 1$, comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{a^2}\right)^n$$

gives

$$\sum_{n=0}^{\infty} |x_{n+1} - x_n| < \infty.$$

(d) We have that

$$\sum_{n=0}^{\infty} (x_{n+1} - x_n)$$

converges absolutely and hence converges. Then

$$\lim_{m\to\infty} (x_m - x_0) = \lim_{m\to\infty} \sum_{n=0}^{m-1} (x_{n+1} - x_n)$$

exists. Let us set

$$c = \lim_{m\to\infty} x_m.$$

This is non-negative as all the $x_n > 0$. We have from the defining relation,

$$\begin{aligned} c &= \lim_{m\to\infty} x_m \\ &= \lim_{m\to\infty} \frac{1}{a + x_{m-1}} = \frac{1}{a+c}, \end{aligned}$$

so

$$c^2 + ac - 1 = 0,$$

so

$$c = \frac{1}{2}\left(-a \pm \sqrt{a^2+4}\right).$$

Here in order that $c \geq 0$, we choose the positive sign of $\sqrt{}$, so

$$\lim_{m\to\infty} x_m = \frac{1}{2}\left(-a + \sqrt{a^2+4}\right).$$

Alternate Solution
Use the contraction mapping theorem.

Question 5. Let $(\Omega, \mathcal{S}, \mu)$ be a sigma finite measure space. For any non negative measurable function on Ω, define the *distribution function* of f, D_f by

$$D_f(t) = \mu(\{\ x\ :\ f(x) > t\ \})\ ,$$

so that D_f is defined on $[0, \infty]$.

Show that for any non negative measurable function f on Ω,

$$\int_\Omega f^2(x)\mathrm{d}\mu = \int_0^\infty \int_0^\infty \min\{\ D_f(s), D_f(t)\}\mathrm{d}s\mathrm{d}t\ .$$

Solution: For $t \geq 0$, define A_t to be the set

$$A_t = \{\ y\ :\ f(y) > t\ \}\ .$$

We write

$$f(x) = \int_0^\infty 1_{A_t}(x)\mathrm{d}t\ .$$

Then, by Fubini (here is where the sigma finiteness comes in),

$$\begin{aligned} \int_\Omega f^2(x)\mathrm{d}x &= \int_\Omega \left(\int_0^\infty 1_{A_t}(x)\mathrm{d}t \int_0^\infty 1_{A_s}(x)\mathrm{d}s \right) \mathrm{d}\mu \\ &= \int_0^\infty \int_0^\infty \left(\int_\Omega 1_{A_t}(x) 1_{A_s}(x)\mathrm{d}\mu \right) \mathrm{d}t\mathrm{d}s \end{aligned}$$

Then since either $A_t \subset A_s$ or $A_s \subset A_t$,

$$\int_\Omega 1_{A_t}(x) 1_{A_s}(x)\mathrm{d}\mu = \min\{\ D_f(s), D_f(t)\}\ .$$

Question 6.
(a) Let μ be a (nonnegative) measure on $[0,1]$ of mass 1, that is $\mu([0,1]) = 1$. Let $f : [0,1] \to \mathbb{R}$ be measurable with respect to μ, and let ϕ be a function convex on the range of f. Prove Jensen's inequality:

$$\phi\left(\int_0^1 f \; d\mu\right) \leq \int_0^1 \phi(f) \; d\mu.$$

(b) Use (a) to prove that if $a_j \in (0,1]$, $j \geq 1$,

$$\sum_{j=1}^{\infty} \frac{\log a_j}{2^j} \leq \log\left(\sum_{j=1}^{\infty} \frac{a_j}{2^j}\right) \tag{1}$$

and

$$\lim_{n\to\infty} \prod_{j=1}^{n} a_j^{1/2^j} \leq \sum_{j=1}^{\infty} \frac{a_j}{2^j}.$$

Solution
(a) As a set of measure 0 does not change the integrals, we assume that f is finite everywhere. Choose

$$-\infty \leq a < b \leq \infty$$

such that ϕ is convex in (a,b) and

$$a < f(\mathbf{x}) < b \text{ for all } \mathbf{x} \in [0,1].$$

Let

$$\gamma = \int_0^1 f \; d\mu.$$

Then as f is integrable with respect to μ, γ is finite and we in fact see from our bounds on f that

$$a < \gamma < b.$$

(The case $b = \infty$ or $a = -\infty$ requires a little more care). Let us consider a line through the point $(\gamma, \phi(\gamma))$ which lies on or below the graph of ϕ throughout (a,b). We can take the slope of this line to be either $D^+\phi(\gamma)$ or $D^-\phi(\gamma)$. Let us call the slope m. Then

$$\phi(\gamma) + m(t - \gamma) \leq \phi(t) \text{ for all } t \in (a,b).$$

Hence, setting $t = f(\mathbf{x})$ for a.e. $\mathbf{x} \in (0,1)$,

$$\phi(\gamma) + m(f(\mathbf{x}) - \gamma) \leq \phi(f(\mathbf{x})).$$

Integrating with respect to $d\mu$ gives

$$\int_0^1 [\phi(\gamma) + m(f(\mathbf{x}) - \gamma)] \, d\mu(\mathbf{x}) \leq \int_0^1 \phi(f(\mathbf{x})) \, d\mu(\mathbf{x})$$

$$\Rightarrow \phi(\gamma) \int_0^1 d\mu + m\left(\int_0^1 f \; d\mu - \gamma \int_0^1 d\mu\right) \leq \int_0^1 \phi(f) \; d\mu$$

Here by the way we defined γ, and as $\int_0^1 d\mu = 1$, we have

$$\int_0^1 f\ d\mu - \gamma \int_0^1 d\mu = 0.$$

So

$$\phi(\gamma) \leq \int_0^1 \phi(f)\ d\mu$$
$$\Rightarrow \phi\left(\int_0^1 f\ d\mu\right) \leq \int_0^1 \phi(f)\ d\mu.$$

(b) We choose μ to be a measure having mass $1/2^j$ at the point $1/j$, $j \geq 1$. Then for any function f defined on $\{1/j : j \geq 1\}$, we have

$$\int_0^1 f\ d\mu = \sum_{j=1}^{\infty} f(1/j)/2^j,$$

at least if $f \geq 0$, or if the series on the right-hand side is absolutely convergent. Note that

$$\mu([0,1]) = \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,$$

so μ fulfills the conditions of (a). Next, if

$$\sum_{j=1}^{\infty} \frac{\log a_j}{2^j} \text{ diverges to } -\infty,$$

then (1) is trivially true. So we assume that

$$\sum_{j=1}^{\infty} \frac{\log a_j}{2^j} > -\infty.$$

As all $\log a_j \leq 0$, this forces

$$\sum_{j=1}^{\infty} \frac{|\log a_j|}{2^j} < \infty.$$

Also, as $a_j \in (0,1]$, so

$$\sum_{j=1}^{\infty} \frac{a_j}{2^j} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

In particular, if we define

$$f(1/j) = \log a_j,\ j \geq 1,$$

then it follows that f is integrable with respect to μ, for

$$\int_0^1 |f|\, d\mu = \sum_{j=1}^{\infty} \frac{|\log a_j|}{2^j} < \infty.$$

We apply (a) with the convex function $\phi(x) = \exp(x)$ to deduce that

$$\exp\left(\int_0^1 f \ d\mu\right) \leq \int_0^1 \exp(f) \ d\mu$$
$$\Rightarrow \exp\left(\sum_{j=1}^{\infty} \frac{\log a_j}{2^j}\right) \leq \sum_{j=1}^{\infty} \frac{a_j}{2^j}.$$

Then (1) follows.

Finally, using continuity of exp,

$$\begin{aligned} \exp\left(\sum_{j=1}^{\infty} \frac{\log a_j}{2^j}\right) &= \exp\left(\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\log a_j}{2^j}\right) \\ &= \lim_{n\to\infty} \exp\left(\sum_{j=1}^{n} \frac{\log a_j}{2^j}\right) \\ &= \lim_{n\to\infty} \exp\left(\log \prod_{j=1}^{n} a_j^{1/2^j}\right) = \lim_{n\to\infty} \prod_{j=1}^{n} a_j^{1/2^j}. \end{aligned}$$

Question 7. Fix any p with $0 < p < 1$. Let $(\Omega, \mathcal{S}, \mu)$ be a sigma finite measure space. Define $L^p(\mu)$ to be the set of *a.e.* equivalence classes of measurable functions f on Ω such that

$$\int_\Omega |f(x)|^p \mathrm{d}\mu < \infty \ .$$

(a) Show that for all $s, t \geq 0$, and $0 < p < 1$,

$$(s+t)^p \leq s^p + t^p \ .$$

(b) Show that any finite linear combination of functions in $L^p(\mu)$ again belongs to $L^p(\mu)$.

(c) Show that

$$d_p(f,g) = \int_\Omega |f(x) - g(x)|^p \mathrm{d}\mu$$

defines a metric on $L^p(\mu)$. Show also that with this metric $L^p(\mu)$ is complete.

Solution: (a)

$$\begin{aligned}(s+t)^p = \int_0^{s+t} pu^{p-1}\mathrm{d}t &= \int_0^s pu^{p-1}\mathrm{d}t + \int_0^t p(u+s)^{p-1}\mathrm{d}t \\ &\leq \int_0^s pu^{p-1}\mathrm{d}t + \int_0^t pu^{p-1}\mathrm{d}t = s^p + t^p \ .\end{aligned}$$

The inequality holds since $u \mapsto u^{p-1}$ is monotone decreasing for $0 < p < 1$.

(b) This is immediate given the inequality from **(a)**.

(c) For f, g and h in $L^p(\mu)$,

$$|f - h| = |(f-g) + (g-h)| \leq |f-g| + |g-h| \ .$$

By the inequality in **(a)**, it is now clear that

$$|f-h|^p \leq |f-g|^p + |g-h|^p \ ,$$

and hence that

$$d_p(f,h) \leq d_p(f,g) + d_p(g,h) \ .$$

This proves the triangle inequality, and the remaining requirement for d_p to be a metric are plainly satisfied.

As for the completeness, let $\{f_n\}$ be a Cauchy sequence. Pass to a subsequence $\{f_{n_k}\}$ such that

$$d_p(f_{n_k}, f_{n_{k+1}}) \leq 2^{-k} \ .$$

Then by the monotone convergence theorem,

$$F = \sum_{k=1}^\infty |f_{n_k} - f_{n_{k+1}}|^p$$

is integrable with $\int_\Omega F \mathrm{d}\mu \leq 1$.

Since $0 < p < 1$, it follows that $\sum_{k=1}^\infty |f_{n_k} - f_{n_{k+1}}|$ converges almost everywhere. Since absolute convergence implies convergence,

$$\sum_{k=1}^\infty (f_{n_k} - f_{n_{k+1}})$$

converges almost everywhere. Let f denote the sum.

Next,

$$|f - f_{n_k}| = \left| \sum_{\ell=k}^\infty (f_{n_\ell} - f_{n_{\ell+1}}) \right|$$

almost everywhere, so once again using (**a**),

$$|f - f_{n_k}|^p \leq \sum_{\ell=k}^\infty |f_{n_\ell} - f_{n_{\ell+1}}|^p \ ,$$

so that

$$d_p(f_{n_k}, f) \leq 2^{-k} \ .$$

Thus the subsequence converges to f. But then since the original sequence is Cauchy, the whole sequence converges to f. This proves the completeness.

Question 8. Let λ denote Lebesgue measure on $[0, 1]$. Let f be any integrable function on $[0, 1]$. For $p > 0$, define f_p by

$$f_p(t) = pt^{p-1} f(t^p) \ .$$

Prove that

$$\lim_{p \to 1} \int_{[0,1]} |f_p(t) - f(t)| \mathrm{d}\lambda = 0 \ .$$

Solution: fix any $\epsilon > 0$. Chose a function g that is continuous on $[0, 1]$, and satisifes $\|f - g\|_1 < \epsilon$. By a simple calculuation, we also have $\|f_p - g_p\|_1 < \epsilon$.

Now by the triangle inequality (Minkowski's inequality),

$$\|f - f_p\|_1 \leq \|f - g\|_1 + \|g - g_p\|_1 + \|g_p - f_p\|_1 \leq \|g - g_p\|_1 + 2\epsilon \ .$$

Now since g is continuous on $[0, 1]$, and therefore bounded, we may use the constant bound in the dominated convergence theorem to conclude that

$$\lim_{p \to 1} \|g - g_p\|_1 = 0 \ .$$

Hence for all p sufficiently close to 1, $\|f - f_p\|_1 \leq 3\epsilon$, which proves the result.