Problem 1. Let $X := (0, \infty)$ and $1 < p < \infty$. For any $f \in \mathcal{L}^p(X, m)$, where $m$ is the Lebesgue measure, define the function

$$F(x) := \frac{1}{x} \int_0^x f(t) \, dt \quad \text{for all } x \in X.$$  

Then prove Hardy’s inequality

$$\|F\| \leq \frac{p}{p-1} \|f\|.$$  

Hint: Assume first that $f \in C_c(X)$ and $f \geq 0$, i.e. $f$ is a continuous and positive function with compact support in $X$. Integration by parts gives

$$\int_0^\infty F^p(x) \, dx = -p \int_0^\infty F^{p-1}(x)xF'(x) \, dx.$$  

Note that $xF'(x) = f(x) - F(x)$.

Solution. Observe first that, since $f \in \mathcal{L}^p(X, m)$, $f \in \mathcal{L}^1((0, \infty), m)$ for every $x \in X$. Thus $F(x)$ is well defined.

Let $f \in C_c(X)$ and $f \geq 0$. Then there exists a number $\delta > 0$ such that $\text{spt } f \subseteq [\delta, \delta^{-1}]$. This implies that $F(x) = 0$ if $x \leq \delta$ and $F(x) = C/x$ for $x \geq \delta^{-1}$, with $C > 0$ some constant. Hence $F(x) \in \mathcal{L}^p(X, m)$.

Since $f$ is continuous, the function $F$ is continuously differentiable, by the fundamental theorem of calculus. Integration by parts then gives

$$\int_0^\infty F^p(x) \, dx = -p \int_0^\infty F^{p-1}(x)xF'(x) \, dx = -p \int_0^\infty F^{p-1}(x)f(x) \, dx + p \int_0^\infty F^p(x) \, dx,$$

since $xF'(x) = f(x) - F(x)$. We used that $F$ vanishes at the boundary. Then

$$\int_0^\infty F^p(x) \, dx \leq \frac{p}{p-1} \int_0^\infty F^{p-1}(x)f(x) \, dx.$$  

Applying Hölder inequality to the right hand side, we obtain

$$\int_0^\infty F^p(x) \, dx \leq \frac{p}{p-1} \left( \int_0^\infty F^p(x) \, dx \right)^{(p-1)/p} \left( \int_0^\infty f^p(x) \, dx \right)^{1/p}.$$  

Then Hardy’s inequality follows because $p$ and $p/(p-1)$ are conjugate exponents.

For general nonnegative $f \in \mathcal{L}^p(X, m)$ consider a sequence of nonnegative functions $f_n \in C_c(X)$ with $f_n \to f$ a.e. monotonically.

By the monotone convergence theorem, we then have

$$F_n(x) := \frac{1}{x} \int_0^x f_n(t) \, dt \to \frac{1}{x} \int_0^x f(t) \, dt = F(x)$$

for all $x \in X$ monotonically.

Hence $F_n^p(x) \to F^p(x)$ and $f_n^p(x) \to f^p(x)$ monotonically.

Applying the monotone convergence theorem again, we conclude that

$$\|F\| = \lim_{n \to \infty} \|F_n\| \leq \frac{p}{p-1} \lim_{n \to \infty} \|f_n\| = \frac{p}{p-1} \|f\|.$$  

If $f$ is real- or complex-valued, then we use that

$$|F(x)| = \frac{1}{x} \int_0^x |f(t)| \, dt \leq \frac{1}{x} \int_0^x |f(t)| \, dt \quad \text{for all } x \in X.$$  

Applying Hardy’s inequality to $|f|$ instead of $f$ we obtain

$$\|F\| \leq \left\| \frac{1}{x} \int_0^x |f(t)| \, dt \right\| \leq \frac{p}{p-1} \|f\| = \frac{p}{p-1} \|f\|.$$
Problem 2. Let $X := [0, 1]$.

1. For any $n \geq 2$ let $A_n$ denote the set of all functions $f \in C(X)$ for which there exists a point $x \in [0, 1 - \frac{1}{n}]$ such that $|f(x + h) - f(x)| \leq nh$ for all $0 < h < \frac{1}{n}$. Show that $A_n$ is nowhere dense in $C(X)$ with the uniform topology.

2. Use the above result to show that the set of functions on $X$ that do not admit left nor right derivatives at any point of $X$ is dense in $C(X)$ in the uniform topology.

Solution. For any $n \geq 2$ let $A_n$ be defined as above. We prove that $A_n$ is closed. Let $\{f_k\} \subset A_n$ such that $\lim_{k \to \infty} \|f - f_k\| = 0$. For every $k$ there exists $x_k$ such that $|f_k(x_k + h) - f_k(x_k)| \leq nh$ for all $0 < h < \frac{1}{n}$. Since $[0, 1 - \frac{1}{n}]$ is compact there exists a subsequence $x_{k_j}$ that converges to a point $x \in [0, 1 - \frac{1}{n}]$. We then have that $\lim_{j \to \infty} f_k(x_{k_j} + h) = f(x + h)$ for every $h$, hence

$$|f(x + h) - f(x)| \leq nh \quad \text{for all } 0 < h < \frac{1}{n},$$

and so $f \in A_n$.

We now prove that $A_n$ is nowhere dense in $C(X)$, by showing that in any $\varepsilon$-neighborhood around any function $f \in A_n$, there exists $\tilde{f} \in C(X) \setminus A_n$. Hence the interior of $A_n$ is empty. To this end, let $g \in C(X)$ be such that $|g(x)| \leq \varepsilon$ and for every $x$ we have $|g(x + h) - g(x)| \geq 3hn$ for $h > 0$ small enough (depending on $x$). For example, let

$$g(x) := \begin{cases} 3nx & 0 \leq x \leq \frac{2\varepsilon}{3n}, \\ -3nx + 2\varepsilon & \frac{2\varepsilon}{3n} \leq x \leq \frac{2\varepsilon}{3} \end{cases},$$

and $g(x) := g(x - \frac{2\varepsilon}{3n})$ for $x \notin [0, \frac{2\varepsilon}{3n}]$.

Given $f \in A_n$, we define $\tilde{f} := f + g$, which implies that $\|\tilde{f} - f\| = \|g\| = \varepsilon$.

But for every $x \in [0, 1 - \frac{1}{n}]$ and $h$ small enough we have

$$|\tilde{f}(x + h) - \tilde{f}(x)| \geq |g(x + h) - g(x)| - |f(x + h) - f(x)| \geq 3nh - nh > nh,$$

so that $\tilde{f} \notin A_n$.

From the above result it follows that for every $n$ the set $A_n^c$ is open and dense in $C(X)$.

From Baire’s category theorem we know that $A = \bigcap_{n=2}^{\infty} A_n^c$ is dense in $C(X)$. Observe that if $f \in A$, then for every $x \in X$ and every $n \geq 2$ there exists a $0 < h < \frac{1}{n}$ such that

$$\left| \frac{f(x + h) - f(x)}{h} \right| > n,$$

so that the right derivative of $f$ at $x$ does not exists.

In a similar fashion, we let $B_n$ denote the set of all functions $f \in C(X)$ for which there exists a point $x \in [\frac{1}{n}, 1]$ such that $|f(x + h) - f(x)| \leq nh$ for all $0 < h < \frac{1}{n}$. Then one can show that $B_n$ is closed and nowhere dense, for all $n \geq 2$. Defining $B := \bigcap_{n=2}^{\infty} B_n^c$, we use again Baire’s category theorem to conclude that $B$ is dense in $C(X)$. If $f \in B$, then for every $x \in X$ the left derivative of $f$ at $x$ does not exists.

Finally the set $C := A \cup B$ is dense and if $f \in C$ then for every $x \in X$ the left and right derivatives of $f$ at $x$ do not exist.
Problem 3. Let $B$ be a Banach space.

(1) Prove that if $T: X \rightarrow X$ is a bounded linear operator with $\|\text{id} - T\| < 1$, where $\text{id}$ is the identity operator on $X$, then $T$ is invertible.

(2) Let $T$ be as before and consider another bounded linear operator $S: X \rightarrow X$ with $\|S - T\| < \|T^{-1}\|^{-1}$. Prove that $S$ is invertible. Show that the set of invertible bounded linear operators from $X$ to itself is an open set in the operator norm topology.

Solution. We write $T = \text{id} - (\text{id} - T)$ and claim that

$$(\text{id} - (\text{id} - T))^{-1} = \sum_{k=0}^{\infty} (\text{id} - T)^k.$$ 

We note first that the series is absolutely convergent because

$$\sum_{k=0}^{\infty} \|\text{id} - T\|^k \leq \sum_{k=0}^{\infty} \|\text{id} - T\|^k,$$

which is finite because $\|\text{id} - T\| < 1$ and so the series is a geometric series. Absolute convergence implies convergence in norm.

For any $N \in \mathbb{N}$ we can now write

$$(\text{id} - (\text{id} - T)) \circ \sum_{k=0}^{N} (\text{id} - T)^k = \text{id} - (\text{id} - T)^{N+1}.$$ 

Sending $N \rightarrow \infty$ and using the fact that

$$\|(\text{id} - T)^{N+1}\| \leq \|\text{id} - T\|^{N+1} \rightarrow 0,$$

again because $\|\text{id} - T\| < 1$, we prove the claim.

Note now that

$$\|S \circ T^{-1} - \text{id}\| = \|(S - T) \circ T^{-1}\| \leq \|S - T\| \|T^{-1}\| < 1,$$

by assumption. Applying (1), we find that the operator $S \circ T^{-1}$ is invertible, so

$$(S \circ T^{-1})^{-1} = T \circ S^{-1}$$

exists. Since $T$ is invertible by assumption, we conclude that $S$ is invertible as well and

$$S^{-1} = T^{-1} \circ (S \circ T^{-1})^{-1}.$$ 

This shows that the set of invertible operators is open because for any invertible operator $T$, the open ball around $T$ with radius less than $\|T^{-1}\|^{-1}$ is also contained in the set of invertible operators.
Problem 4. Let \( X \) be a compact metric space and \( \mathcal{A} \) a closed algebra of continuous real-valued functions separating points in \( X \).

1. Show that if \( f \in \mathcal{A} \) then \( \sqrt{|f|} \in \mathcal{A} \).
2. Show that \( \mathcal{A} \) is either all of \( C(X) \) or there exists a point \( x \in X \) such that
   \[
   \mathcal{A} = \{ f \in C(X) \mid f(x) = 0 \}.
   \]

**Hint:** Prove first that there exists a sequence of polynomials \( \{P_n\}_{n \in \mathbb{N}} \) such that \( P_n(z) \rightarrow \sqrt{z} \) uniformly on \([\varepsilon, 1]\) for every \( \varepsilon > 0 \). To this end, consider the iteration

\[
P_1(z) := 0 \quad \text{and} \quad P_{n+1}(z) := P_n(z) + \frac{1}{2}(z - P_n(z)^2) \quad \text{for all } z \in [0, 1], n \in \mathbb{N}.
\]

**Solution.** Let the polynomials \( P_n \) be defined as above and note that by induction we have that \( P_n(0) = 0 \) for all \( n \in \mathbb{N} \). Observe now that for all \( z \in [0, 1] \) it holds

\[
z - P_{n+1}(z)^2 = (z - P_n(z)^2) \left( 1 - P_n(z) - \frac{1}{4}(z - P_n(z)^2) \right)
\]

and that if \( 0 \leq P_n(z) \leq \sqrt{z} \) then

\[
0 \leq 1 - \sqrt{z} \leq 1 - P_n(z) - \frac{1}{4}(z - P_n(z)^2) \leq 1 - \frac{1}{4}z
\]

This easily implies that \( z - P_{n+1}(z)^2 \geq 0 \) (since it is the product of two positive quantities) and thus \( P_{n+1}(z) \geq P_n(z) \geq 0 \). By induction again we have that \( P_n(z) \) is an increasing sequence with \( 0 \leq P_n(z) \leq \sqrt{z} \) for every \( n \) and that \( P_n(z) \) converges uniformly to \( \sqrt{z} \) in \( C([\varepsilon, 1]) \) for every \( \varepsilon \).

Since \( \sqrt{z} \) and \( P_n(z) \) are continuous functions the convergence is uniform in \( C([0, 1]) \).

Given \( f \in \mathcal{A} \) set \( c = \|f\|_{\infty} \). We have that \( P_n(f^2/c^2) \in \mathcal{A} \) for every \( n \). Moreover, since \( 0 \leq f^2/c^2 \leq 1 \), \( P_n(f^2/c^2) \) converges uniformly to \( |f|/c \). Hence \( |f| \in \mathcal{A} \). By the same argument we get that \( P_n(|f|/c) \) converges uniformly to \( \sqrt{|f|/c} \) and thus \( \sqrt{|f|} \in \mathcal{A} \).

(2) Suppose that there is no \( x \in X \) such that \( f(x) = 0 \) for all \( f \in \mathcal{A} \). Thus for every \( x \in X \) we can find \( f_x \in \mathcal{A} \) such that \( f_x(x) \neq 0 \). Since \( f_x \) is continuous there exists an open neighborhood \( U_x \) of \( x \) such that \( f_x(y) \neq 0 \) for all \( y \in U_x \).

Observe that \( \{U_x\}_{x \in X} \) is an open covering of \( X \). Since \( X \) is compact, there exists a finite sub-covering \( U_{x_i}, i = 1, \ldots, n \). Consider now the function \( g = \sum_{i=1}^n f_{x_i} \). Clearly \( g \in \mathcal{A} \) and \( g(x) > 0 \) for all \( x \in X \).

Dividing \( g \) by its norm, we may assume that \( g(x) \leq 1 \) for every \( x \in X \). Moreover, since \( X \) is compact there exists a constant \( \varepsilon > 0 \) with \( \varepsilon \leq g(x) \) for all \( x \in X \).

We now define the functions \( h_N(x) := \varepsilon \sqrt{g(x)} \) for all \( x \in X \) and \( N \in \mathbb{N} \), which is in \( \mathcal{A} \) by (1).

We have that \( \lim_{n \to \infty} h_N(x) = 1 \) for every \( x \in X \). In fact, the convergence is uniform because \( g(x) \geq \varepsilon > 0 \). Since \( \mathcal{A} \) is closed, we conclude that the constant functions are in \( \mathcal{A} \).

Thus \( \mathcal{A} \) is an algebra of continuous function that separates points and contains the constant functions. Therefore, by the Stone-Weierstrass theorem, we have that \( \mathcal{A} = C(X) \).

On the other hand, if there exists \( x \in X \) such that \( f(x) = 0 \) for all \( f \in \mathcal{A} \), then for every \( y \neq x \) there exists \( f_y \in \mathcal{A} \) with \( f_y(y) \neq 0 \) because \( \mathcal{A} \) separates points. That is, there can at most exist one point in \( X \) at which all functions in \( \mathcal{A} \) vanish. The same argument as before then implies that \( \mathcal{A} \) contains all continuous functions \( f \) with \( f(x) = 0 \).
Problem 5. Let \( X \) be some set and \( \mathcal{P}(X) \) its power set. Consider a map \( K: \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) with the following properties:

1. \( K(\emptyset) = \emptyset \);
2. \( A \subset K(A) \) for all \( A \);
3. \( K(K(A)) = K(A) \) for all \( A \);
4. \( K(A \cup B) = K(A) \cup K(B) \) for all \( A, B \),

and let \( \mathcal{F} := \{ A \subseteq X : K(A) = A \} \). Prove that \( \emptyset, X \in \mathcal{F} \), and that \( \mathcal{F} \) is closed under arbitrary intersections and finite unions. It follows that \( \mathcal{T} := \{ U \subseteq X : U^c \in \mathcal{F} \} \) is a topology. Prove that for every \( A \), the set \( K(A) \) is the closure of \( A \) with respect to the topology \( \mathcal{T} \).

Solution. Since \( A \subseteq B \) we have \( B = A \cup B \). Then (4) implies

\[
K(B) = K(A \cup B) = K(A) \cup K(B) \supset K(A).
\]

By (1), we have \( K(\emptyset) = \emptyset \) and thus \( \emptyset \in \mathcal{F} \).

By (2), we have \( X \subset K(X) \) and clearly \( K(X) \subset X \). Hence \( X \in \mathcal{F} \).

Let \( A_\lambda \in \mathcal{F}, \lambda \in \Lambda \), be an arbitrary collection of sets. Then \( \bigcap_{\lambda \in \Lambda} A_\lambda \subset A_\beta \) for all \( \beta \). Therefore

\[
K\left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \subset K(A_\beta)
\]

for all \( \beta \), and so

\[
K\left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \subset \bigcap_{\beta \in \Lambda} K(A_\beta) = \bigcap_{\beta \in \Lambda} A_\beta \subset K\left( \bigcap_{\beta \in \Lambda} A_\beta \right).
\]

The equality follows from the assumption \( K(A_\beta) = A_\beta \) for all \( \beta \), and the last inclusion from (2).

We conclude that all terms are in fact equal, hence \( \mathcal{F} \) is closed under arbitrary intersections.

Let now \( A_1, A_2 \in \mathcal{F} \) be given. From (4) we obtain

\[
K(A_1 \cup A_2) = K(A_1) \cup K(A_2) = A_1 \cup A_2.
\]

The last equality follows from the assumption that \( K(A_i) = A_i \) for \( i = 1, 2 \). By induction, we have that \( \mathcal{F} \) is closed under finite unions.

The collection \( \mathcal{F} \) contains all closed sets of the topology \( \mathcal{T} \) because \( B \) being closed is, by definition, equivalent to \( B^c \) being open, that is \( B^c \in \mathcal{F} \). This in turn is equivalent to \( (B^c)^c = B \in \mathcal{F} \).

Note that for any \( A \), the set \( K(A) \in \mathcal{F} \) because of (3).

Let now \( \bar{A} \) be the closure of \( A \) in the topology \( \mathcal{T} \). Since \( \bar{A} \subset K(A) \) by (2), and since \( K(A) \in \mathcal{F} \), we have \( \bar{A} \subset K(A) \).

On the other hand, since \( \mathcal{F} \) is closed under arbitrary intersections, we have \( \bar{A} \in \mathcal{F} \). Then \( A \subset \bar{A} \) implies

\[
K(A) \subset K(\bar{A}) = \bar{A} \subset K(A),
\]

and so all sets are in fact equal.
Problem 6. Let \((X, \mathcal{M}, \mu)\) be a measure space and consider functions \(f \in L^p(\mu)\) and \(g \in L^q(\mu)\) with \(1 < p < \infty\) and \(1/p + 1/q = 1\). Show that \(\int_X |fg| d\mu = \|f\|_p \|g\|_q\) if and only if there exist constants \(C_1, C_2 \geq 0\), not both equal to zero, such that \(C_1|f|^p = C_2|g|^q\).

Hint: Show first that for all \(a, b \geq 0\) and \(t \in (0, 1)\) we have \(a^{t+b-1-t} \leq ta + (1-t)b\), with equality holding if and only if \(b = a\). To this end, consider \(h(x) := 1 - t + tx - xt\) for \(x \in [0, 1]\).

Solution. Let \(h\) be defined as above and note that \(h'(x) = t(1 - x^{t-1}) < 0\) for all \(x \in [0, 1]\) and \(t \in (0, 1)\). Since \(h(1) = 0\) we conclude that \(h(x) > 0\) for all \(x \in [0, 1]\). With \(x := a/b\) we find
\[
a^{t+b-1-t} \leq ta + (1-t)b,
\]
and equality holds if and only if \(a = b\). If \(b = 0\), then there is nothing to prove.

Assume now that \(C_1|f|^p = C_2|g|^q\) with \(C_1 > 0\). We have
\[
\int_X |fg| d\mu = \left( \frac{C_2}{C_1} \right)^{\frac{1}{p}} \int_X |g|^q d\mu = \left[ \int_X \left( \frac{C_2}{C_1} \right) |g|^q d\mu \right]^{\frac{1}{p}} \left[ \int_X |g|^q d\mu \right]^{\frac{1}{q}} = \left[ \int_X |f|^p d\mu \right]^{\frac{1}{p}} \left[ \int_X |g|^q d\mu \right]^{\frac{1}{q}} = \|f\|_p \|g\|_q.
\]
If \(C_1 = 0\), then \(g = 0\) and the identity follows trivially.

For the converse direction, assume that \(\int_X |fg| d\mu = \|f\|_p \|g\|_q\). This identity is certainly satisfied if \(f = 0\), in which case we may take \(C_1 = 1\) and \(C_2 = 0\) (and thus \(g\) is arbitrary). In a similar way, we may argue if \(g = 0\).

Assume therefore that neither \(f\) or \(g\) are the zero function. Now fix \(x \in X\) and let
\[
t := \frac{1}{p}, \quad a := \left( \frac{|f(x)|}{\|f\|_p} \right)^p, \quad b := \left( \frac{|g(x)|}{\|g\|_q} \right)^q,
\]
which are well-defined since \(\|f\|_p \neq 0\) and \(\|g\|_q \neq 0\). From inequality (1) we obtain
\[
\frac{1}{p} \left( \frac{|f(x)|}{\|f\|_p} \right)^p + \frac{1}{q} \left( \frac{|g(x)|}{\|g\|_q} \right)^q - \frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq 0.
\]
Integrating we get
\[
\int_X \left[ \frac{1}{p} \left( \frac{|f(x)|}{\|f\|_p} \right)^p + \frac{1}{q} \left( \frac{|g(x)|}{\|g\|_q} \right)^q - \frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \right] d\mu = \frac{1}{p} + \frac{1}{q} - \frac{\int_X |f(x)g(x)| d\mu}{\|f\|_p \|g\|_q} = 0,
\]
where the last equality follows from our hypothesis. This implies that
\[
\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} = \frac{1}{p} \left( \frac{|f(x)|}{\|f\|_p} \right)^p + \frac{1}{q} \left( \frac{|g(x)|}{\|g\|_q} \right)^q
\]
for \(\mu\text{-a.e.} \ x \in X\). Since equality holds in (1) if and only if \(a = b\) we get that
\[
\left( \frac{|f(x)|}{\|f\|_p} \right)^p = \left( \frac{|g(x)|}{\|g\|_q} \right)^q
\]
for \(\mu\text{-a.e.} \ x \in X\). It is now enough to take \(C_1 := \|g\|_q^q\) and \(C_2 := \|f\|_p^p\) to obtain the thesis.
Problem 7. Let \((X, \mathcal{M}, \mu)\) be a measure space. For any given set \(E \in \mathcal{M}\) we denote by \(L^2(E, \mu)\) the subspace of \(L^2(X, \mu)\) of functions that vanish in \(X \setminus E\). Let \(\{E_n\}\) be a sequence of pairwise disjoint sets \(E_n \in \mathcal{M}\) with \(X = \bigcup_{n=1}^{\infty} E_n\).

Prove that \(\{L^2(E_n, \mu)\}\) is a sequence of mutually orthogonal subspaces of \(L^2(X, \mu)\), and that every \(f \in L^2(X, \mu)\) can be written uniquely as \(f = \sum_{n=1}^{\infty} f_n\) with \(f_n \in L^2(E_n, \mu)\) (prove that the series converges in norm).

Solution. Let \(f_m \in L^2(E_m, \mu)\) and \(f_n \in L^2(E_n, \mu)\) with \(m \neq n\). Since \(E_m \cap E_n = \emptyset\), we find that \(f_m\) vanishes on \(E_n \subset X \setminus E_m^c\) and \(f_n\) vanishes on \(E_m \subset X \setminus E_n^c\). Therefore we have

\[
\int_X f_m f_n \, d\mu = 0,
\]

so the two subspaces are mutually orthogonal with respect to the inner product.

Let now \(f \in L^2(X, \mu)\) be given and define

\[
f_n := f \chi_{E_n} \in L^2(E_n, \mu) \quad \text{for all } n \in \mathbb{N}.
\]

Then the functions \(f_n\) are mutually orthogonal as shown above.

Since \(f \in L^2(X, \mu)\), we have \(|f|^2 \in L^1(X, \mu)\), and so \(|f|^2 \, d\mu\) is a finite measure. Define \(F_N := \bigcup_{n=1}^{N} E_n\) for all \(N \in \mathbb{N}\). Then \(\{F_N\}\) is a monotone increasing sequence of measurable sets, with \(X = \bigcup_{N=1}^{\infty} F_N\).

For any \(N\) we now write

\[
\sum_{n=1}^{N} \int_X |f_n|^2 \, d\mu = \sum_{n=1}^{N} \int_{E_n} |f|^2 \, d\mu = \int_{F_N} |f|^2 \, d\mu.
\]

Sending \(N \to \infty\) and using continuity from below, we obtain

\[
\sum_{n=1}^{\infty} \int_X |f_n|^2 \, d\mu = \lim_{N \to \infty} \int_{F_N} |f|^2 \, d\mu = \int_X |f|^2 \, d\mu,
\]

which is finite.

Let now \(M, N \in \mathbb{N}\) be given an assume without loss of generality that \(N \leq M\). Then

\[
\left\| \sum_{n=1}^{M} f_n - \sum_{n=1}^{N} f_n \right\|^2 = \left\| \sum_{n=N+1}^{M} f_n \right\|^2 = \int_X \left| \sum_{n=N+1}^{M} f_n \right|^2 \, d\mu = \sum_{n=N+1}^{M} \int_X |f_n|^2 \, d\mu,
\]

which converges to zero as \(M, N \to \infty\). In the last equality we used the orthogonality of the \(f_n\). Using the Cauchy criterion, we conclude that the series \(\sum_{n=1}^{\infty} f_n\) converges in norm.

Multiplying the identity \(f = \sum_{n=1}^{\infty} f_n\) by \(\chi_{E_m}\) for some \(m \in \mathbb{N}\), we find

\[
f \chi_{E_m} = \left( \sum_{n=1}^{\infty} f_n \right) \chi_{E_m} = f_m \chi_{E_m} = f_m,
\]

so that is the only way to define the functions \(f_m\).
Problem 8. Let \((X, \mathcal{M}, \mu)\) be a measure space. Show that \(f : X \rightarrow [0, \infty)\) is measurable if and only if there exist nonnegative constants \(\{c_n\}_{n=0}^{\infty}\) and measurable sets \(\{E_n\}_{n=0}^{\infty}\) such that
\[
f(x) = \sum_{n=0}^{\infty} c_n \chi_{E_n}(x).
\]

Solution. If (2) holds, then
\[
f(x) = \lim_{N \to \infty} f_N(x),
\]
with functions \(f_N\) defined by
\[
f_N(x) := \sum_{n=0}^{N} c_n \chi_{E_n}(x) \quad \text{for all } x \in X.
\]
The function \(f\) is therefore the pointwise limit of the sequence measurable functions \(f_N\), and so \(f\) is measurable as well.

For the converse direction, we use that for any measurable function \(g : X \rightarrow [0, \infty)\) and any \(N \in \mathbb{N}\), the simple function
\[
\phi(x) := \sum_{i=0}^{\lfloor 2^{-N} \rfloor} 2^{-n} \chi_{F_n}, \quad \text{where} \quad F_n := g^{-1}\left(\left[n, n + 1\right]\right)
\]
satisfies \(0 \leq \phi(x) \leq g(x)\) for all \(x \in X\) and \(g(x) - \phi(x) \leq 2^{-N}\) for all \(x \in g^{-1}([0, 2^N])\).

Starting from our function \(f\) we construct the simple function \(\phi_1\) such that \(0 \leq f(x) - \phi_1(x) \leq 2^{-1}\) for all \(x \in f^{-1}([0, 2])\).

Now observe that the function \(f_1 := f - \phi_1\) is again nonnegative and measurable, so we can find a simple function \(\phi_2\) with \(0 \leq f_1(x) - \phi_2(x) \leq 2^{-2}\) for all \(x \in f_1^{-1}([0, 2^2])\).

We define recursively a sequence of simple functions \(\phi_n\) and measurable functions \(f_n\) such that
\[
\begin{array}{l}
(1) \quad f_0 := f; \\
(2) \quad \phi_n \text{ is such that } 0 \leq f_{n-1}(x) - \phi_n(x) \leq 2^{-n} \text{ for every } x \in f_{n-1}^{-1}([0, 2^n]); \\
(3) \quad f_n := f_{n-1} - \phi_n.
\end{array}
\]

Since \(0 \leq f_n(x) \leq f(x)\) we have that for every \(x \in f^{-1}([0, 2^n])\) it holds
\[
0 \leq f(x) - \sum_{i=0}^{n} \phi_i(x) \leq 2^{-n}.
\]

Since the function \(f\) takes only finite values, this implies that for all \(x \in X\) we have
\[
f(x) = \sum_{n=0}^{\infty} \phi_n(x).
\]

If we now write
\[
\phi_n(x) := \sum_{i=0}^{N_n} c_{n,i} \chi_{E_{n,i}}
\]

for suitable nonnegative constants \(c_{n,i}\) and measurable sets \(E_{n,i}\), then we obtain that
\[
f(x) = \sum_{n=0}^{\infty} \sum_{i=0}^{N_n} c_{n,i} \chi_{E_{n,i}}.
\]

The thesis follows then by renaming the sets \(E_{n,i}\) and the numbers \(c_{n,i}\).