Analysis Comprehensive Exam
Spring 2010

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of nonnegative measurable functions on X, such that $f_n \to f$ pointwise.

   (a) Show that if $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu < \infty$ then $\lim_{n\to\infty} \int_E f_n = \int_E f$ for all $E \in \mathcal{M}$.

   (b) Find an example on $\mathbb{R}$ (with Lebesgue measure) which shows that the statement above is not always true if $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu = \infty$.

Solution:

(a) By Fatou's lemma, for every $E \in \mathcal{M}$, we have

$$\int_E f d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu.$$

Using this inequality for E and $X \setminus E$ we obtain:

$$\begin{aligned}
\int_E f d\mu &\le \liminf_{n\to\infty} \int_E f_n d\mu \le \limsup_{n\to\infty} \int_E f_n d\mu \\
&= \limsup_{n\to\infty} \left[\int_X f_n d\mu - \int_{X\setminus E} f_n d\mu \right] \\
&= \lim_{n\to\infty} \int_X f_n d\mu - \liminf_{n\to\infty} \int_{X\setminus E} f_n d\mu \\
&\le \int_X f d\mu - \int_{X\setminus E} f d\mu \\
&\le \int_E f d\mu,
\end{aligned}$$

which shows that we must have equalities everywhere, completing the proof of (a).

(b) If we take $f_n(x) = \chi_{(-\infty,0)}(x) + \frac{\chi_{(0,n)}(x)}{n}$, then $f_n(x) \to f(x) = \chi_{(-\infty,0)}(x)$ pointwise (even uniformly) on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx = \infty$.
However, $\int_0^\infty f_n(x)dx = 1 \neq 0 = \int_0^\infty f(x)dx$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_1, f_2, \dots$ and f be measurable complex-valued functions on X such that $f_n \to f$ a.e. Suppose that there exists a nonnegative measurable function g such that $|f_n| \le g$ and for all $\epsilon > 0$, we have $\mu(\{x \in X : g(x) > \epsilon\}) < \infty$. Prove that $f_n \to f$ almost uniformly, that is for all $\epsilon > 0$ there is a measurable set $E \subset X$ with $\mu(E) < \epsilon$ and f_n converges uniformly to f on $X \setminus E$.

Solution:

The proof is similar to the proof of Egoroff's theorem. For $k, n \in \mathbb{N}$ let

$$E_n(k) = \bigcup_{m=n}^{\infty} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \right\}.$$

Note that $\mu(E_1(k)) < \infty$. Indeed, we have $|f_m(x) - f(x)| \leq 2g$ for all m and almost every x. Since $A = \{x : 2g(x) \geq \frac{1}{k}\}$ has finite measure and $E_1(k) \subset A$ we conclude that $\mu(E_1(k)) < \infty$.

Clearly, for fixed k, $E_n(k)$ decreases as n increases. Since $\mu\left(\cap_{n=1}^{\infty} E_n(k)\right) = 0$ and $\mu(E_1(k)) < \infty$ we conclude that $\mu(E_n(k)) \to 0$ as $n \to \infty$. Given $\epsilon > 0$ and $k \in \mathbb{N}$, choose n_k such that $\mu(E_{n_k}(k)) < \frac{\epsilon}{2^k}$ and let $E = \cup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) < \epsilon$ and we have $|f_n(x) - f(x)| < \frac{1}{k}$ for $n > n_k$ and $x \notin E$. Thus $f_n \to f$ uniformly on $X \setminus E$.

3. Let ν be a σ-finite signed measure and μ a σ-finite positive measure on a measurable space $(X, \mathcal{M})$. Show that the following statement are equivalent:

(a) $|\nu(E)| \leq \mu(E)$ for every $E \in \mathcal{M}$;

(b) $|\nu|(E) \leq \mu(E)$ for every $E \in \mathcal{M}$;

(c) $\nu \ll \mu$ and $|\frac{d\nu}{d\mu}(E)| \leq 1$ for μ-almost every $x \in X$.

Solution: We prove below that (a) implies (b), (b) implies (c), and (c) implies (a).

(a) implies (b). Let $X = P \cup N$ be a Jordan decomposition of X. If $E \in \mathcal{M}$ then

$$\begin{aligned} |\nu|(E) &= |\nu|(E \cap P) + |\nu|(E \cap N) = \nu^+(E \cap P) + \nu^-(E \cap N) = \nu(E \cap P) - \nu(E \cap N) \\ &\leq \mu(E \cap P) + \mu(E \cap N) = \mu(E). \end{aligned}$$

(b) implies (c). If $\mu(E) = 0$ then $|\nu|(E) = 0$ hence $\nu(E) = 0$, which shows that $\nu \ll \mu$. Thus we have

$$d\nu = \frac{d\nu}{d\mu} d\mu \text{ and therefore } d|\nu| = \left| \frac{d\nu}{d\mu} \right| d\mu.$$

Note that if $A \in \mathcal{M}$ is such that $\mu(A) < \infty$ then for every $\epsilon > 0$ the set $A_\epsilon = \left\{ x \in A : \left| \frac{d\nu}{d\mu} \right| \geq 1 + \epsilon \right\}$ is a μ-null set. Indeed we have $|\nu|(A_\epsilon) \geq (1+\epsilon)\mu(A_\epsilon) \geq (1+\epsilon)|\nu|(A_\epsilon)$, leading to $|\nu|(A_\epsilon) = \mu(A_\epsilon) = 0$.

Since μ is σ-finite we have $X = \cup_{k\in\mathbb{N}} A^k$, where $\mu(A^k) < \infty$ for every $k \in \mathbb{N}$. Thus for every $k, n \in \mathbb{N}$ the set $A^k_{1/n} = \left\{x \in A^k : \left|\frac{d\nu}{d\mu}\right| \geq 1 + \frac{1}{n}\right\}$ is μ-null. Hence $\{x : \left|\frac{d\nu}{d\mu}\right| > 1\} = \bigcup_{k,n\in\mathbb{N}} A^k_{1/n}$ is a μ-null set.

(c) implies (a). Since $d\nu = \frac{d\nu}{d\mu} d\mu$ for every $E \in \mathcal{M}$ we have

$$|\nu(E)| = \left|\int_E \frac{d\nu}{d\mu} d\mu\right| \leq \int_E \left|\frac{d\nu}{d\mu}\right| d\mu \leq \int_E d\mu = \mu(E).$$

4. Let H be a separable infinite dimensional Hilbert space and let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H. Show that if $\{v_n\}_{n\in\mathbb{N}}$ is an orthonormal set in H such that $\sum_n ||u_n - v_n||^2 < \infty$ then it is also an orthonormal basis for H. (Hint: Consider first the case when $\sum_n ||u_n - v_n||^2 < 1$)

Solution: Suppose first that $\sum_n ||u_n - v_n||^2 < 1$. We want to show that if $\langle x, v_n\rangle = 0$ for all n, then $x = 0$. Using the Parseval's identity and Schwarz inequality we find:

$$||x||^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle x, u_n - v_n\rangle|^2 \leq ||x||^2 \sum_{n=1}^{\infty} ||u_n - v_n||^2,$$

proving that $x = 0$.

For the general case, we choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} ||u_n - v_n||^2 < 1$. Let

$$u'_n = u_n - \sum_{k=N+1}^{\infty} \langle u_n, v_k\rangle v_k$$

and let us denote by S the linear span of $\{u'_1, u'_2 \ldots, u'_N\}$. Then $H = S \oplus S^{\perp}$. Note that $v_k \in S^{\perp}$ for every $k > N$, and using the same argument we can deduce that $\{v_k : k > N\}$ is a an orthonormal basis for $S^{\perp}$. Indeed, if $x \in S^{\perp}$ and $\langle x, v_k\rangle = 0$ for every $k > N$ then $\langle x, u_k\rangle = 0$ for $k \leq N$. Thus we have

$$||x||^2 = \sum_{n=N+1}^{\infty} |\langle x, u_n\rangle|^2 = \sum_{n=N+1}^{\infty} |\langle x, u_n - v_n\rangle|^2 \leq ||x||^2 \sum_{n=N+1}^{\infty} ||u_n - v_n||^2,$$

showing that $x = 0$ and therefore $\{v_k : k > N\}$ is a an orthonormal basis for $S^{\perp}$. In particular it follows that $v_k \perp S^{\perp}$ for $k \leq N$, so $v_k \in S$ for $k \leq N$. Since S has dimension at most N, we conclude that $v_1, \ldots, v_N$ form an orthonormal basis for S, completing the proof.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, f_n \in L^p$, where $1 \leq p < \infty$. Prove that if $f_n \to f$ a.e., then $||f_n - f||_p \to 0$ if and only if $||f_n||_p \to ||f||_p$.

Solution: The "only if" part follows immediately from Minkowski's inequality since

$$|||f_n||_p - ||f||_p| \leq ||f - f_n||_p.$$

To prove the "if" part denote $F_n = |f_n - f|^p$ and $G_n = 2^p(|f_n|^p + |f|^p)$. Then $F_n \to 0$ a.e., $G_n \to G = 2^{p+1}|f|^p$ a.e.. Moreover we have $F_n, G_n, G \in L^1$, $F_n \leq G_n$ and $\int G_n d\mu \to \int G d\mu$.

Applying Fatou's lemma we find

$$\int G d\mu = \int \lim_{n\to\infty} (G_n - F_n) d\mu \leq \liminf_{n\to\infty} \int (G_n - F_n) d\mu = \int G d\mu - \limsup_{n\to\infty} \int F_n d\mu.$$

Since $\int G d\mu < \infty$, we can subtract it from both sides to get

$$0 \leq \limsup_{n\to\infty} \int F_n d\mu \leq 0,$$

and hence $||f_n - f||_p = \left(\int F_n d\mu\right)^{\frac{1}{p}} \to 0$.

6. Let $E \subset [0, 1]$ be a measurable subset with $|E| > 0$. Let χ denote its characteristic function.
   (a) Show that the function below is continuous function of x.

$$F(x) = \int_{[0,1]} \chi(x - t)\chi(t)\, dt$$

   (b) Show that the set $E + E = \{x + y \;:\; x, y \in E\}$ contains a non-empty interval.

Solution:

(a) Fix $0 < x < 1$ and let x_n be a sequence in $[0, 1]$ with $x_n \to x$. Then, we have

$$\chi(x_n - t)\chi(t) \to \chi(x - t)\chi(t)$$

for all choices of t for which $x - t$ is a Lebesgue point of χ. Almost every $x - t$ is a Lebesgue point, so we conclude that $\chi(x_n - t)\chi(t) \to \chi(x - t)\chi(t)$ almost every where on $t \in [0, 1]$. All functions are bounded by one, and we are on a finite measure space, so by the Bounded Convergence Theorem,

$$F(x_n) = \int_{[0,1]} \chi(x_n - t)\chi(t)\, dt \longrightarrow F(x)$$

(b) Since $\chi(x-t)\chi(t)$ is a nonnegative measurable function on $\mathbb{R}^2$ we can apply Tonelli's theorem to deduce

$$\int_{\mathbb{R}} F(x)dx = \int_{[0,1]} \left[\int_{\mathbb{R}} \chi(x-t)dx\right] \chi(t)dt = \mu(E)^2 > 0.$$

Hence $F(x)$ is positive on some nonempty interval I. Note that $I \subset E+E$ completing the proof.

7. Let $I \subset [0,1]$ denote a closed interval of positive length. Say that $f : I \longrightarrow \mathbb{R}$ is Lipschitz on I if for some constant C and all $x, y \in I$ we have $|f(x)-f(y)| \leq C|x-y|$. Show that there is a continuous function $f : [0,1] \longrightarrow \mathbb{R}$ that is *not* Lipschitz on any closed interval $I \subset [0,1]$.

Solution: While it is possible to write down such a function in closed form, it is simpler to use the Baire Category Theorem. In so doing, a standard issue arises, that there are an uncountable number of closed intervals $I \subset [0,1]$. But it suffices to demonstrate that there is a continuous function which is not Lipschitz on any closed interval I with *rational endpoints.* The latter intervals are countable, and we consider an enumeration of them $\{I_k : k \in \mathbb{N}\}$.

The space $C[0,1]$ is a complete metric space, due to the Arzela-Ascoli Theorem. For integers k, let B_k denote those functions $f \in C[0,1]$ for which f is Lipschitz on I_k. If we show that each B_k has empty interior, with respect to the sup-norm topology, we conclude from the Baire Category Theorem that the set

$$C[0,1] \backslash \bigcup_{k \in \mathbb{N}} B_k$$

is non-empty, for otherwise the complete metric space $C[0,1]$ would be the countable union of nowhere dense sets.

Consider a function on $[0,1]$ given by

$$\phi(x) = \sqrt{\min(x, 1-x)}\,.$$

We extend ϕ to all of $\mathbb{R}$ by setting $\phi(x) = 0$ for $x \in \mathbb{R}\backslash[0,1]$. The basic fact is that ϕ is *not Lipschitz* on $[0,1]$. Indeed, it suffices to take $0 < \epsilon < \frac{1}{2}$, and note that $\phi(\epsilon^2) - \phi(0) = \epsilon$. This shows that the Lipschitz constant of ϕ would have to be at least ϵ^{-1}, proving the basic fact.

For an interval I, let us set $\phi_I(x) = \phi((x-c_I)/|I|)$ where c_I is the center of I. As the map $x \to (x-c_I)/|I|$ is itself Lipschitz, it follows from the basic fact that ϕ_I is not Lipschitz on I. Therefore, for $f \in B_k$, and arbitrary $\epsilon > 0$, we have $f + \epsilon\phi_{I_k} \notin B_k$, showing that B_k has empty interior.