1. Let $E \subset (-\pi, \pi)$ with Lebesgue measure $m(E) > 0$. Show that for every $\delta > 0$, there are at most finitely many positive integers n such that $\sin(nx) \geq \delta$ for every $x \in E$.

Solution: Suppose to the contrary that there exist $\delta > 0$ and a strictly increasing sequence of positive integers $\{n_k\}$ such that $\sin(n_k x) \geq \delta$ for all $x \in E$. Consider

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{k} \sin(n_k x).$$

Since $\sum_{k=1}^{\infty} 1/k^2 < \infty$, we have $f \in \mathscr{L}^2([-\pi, \pi])$. Therefore

$$m(\{x \in (-\pi, \pi) : |f(x)| = \infty\}) = 0.$$

However $\sum_{k=1}^{\infty} 1/k = \infty$, so for every $x \in E$, $f(x) = \infty$. This contradicts the fact that $f \in \mathscr{L}^2([-\pi, \pi])$ as $m(E) > 0$.

2. Let X be a topological vector space and $A, B \subset X$. Recall that

$$A + B = \{x + y : x \in A,\ y \in B\}.$$

Show that

(a) if A and B are compact, then $A + B$ is compact;

(b) if A is compact and B is closed, then $A + B$ is closed;

(c) $cl(A) + cl(B) \subset cl(A + B)$, and give an example when the inclusion is strict.

Hint: Think about an example in a Hilbert space.

Solution: (a) Let $\{x_n + y_n\}$ be a sequence in $A + B$ with $x_n \in A$ and $y_n \in B$. A is compact implies that there exists a subsequence $\{x_{n_k}\}$ converging to some x in A. The set B is compact implies that the sequence $\{y_{n_k}\}$ also has a convergent subsequence in B, namely $\{y_{n_{k_j}}\}$ converges to some $y \in B$. Therefore, the subsequence $\{x_{n_{k_j}} + y_{n_{k_j}}\}$ converges in $A + B$ and therefore $A + B$ is compact.

(b) Now suppose A is compact and B is closed. Let $\{x_n + y_n\}$ be a sequence in $A + B$ such that $x_n \in A$ and $y_n \in B$ and $x_n + y_n \to z$ for some $z \in X$ as $n \to \infty$. The set A is compact implies that there exists a subsequence $\{x_{n_k}\}$ converges to some x in A. Therefore $x_{n_k} + y_{n_k} \to z$ and so $y_{n_k} \to z - x$. But B is closed, therefore $z - x \in B$ and so $z \in A + B$. Hense $A + B$ is closed.

(c) Consider $X = l_2 := \{(x_1, x_2, \dots) : \sum |x_n|^2 < \infty\}$, with the standard basis $\{e_n\}$, where $e_n = (\delta_{1n}, \delta_{2n}, \dots)$ and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Let

$$\mathscr{M} = cl(span\{e_{2n} : n = 1, 2, \dots\})$$

and

$$\mathscr{N} = cl(span\{\frac{1}{n}e_{2n-1} + e_{2n} : n = 1, 2, \dots\}).$$

Notice that $\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1} \in cl(\mathscr{M} + \mathscr{N})$, but $\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1}$ is not in $\mathscr{M} + \mathscr{N}$. Indeed, it is easy to see that $e_{2n-1} \in \mathscr{M} + \mathscr{N}$ for each $n = 1, 2, \dots$, therefore $\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1} \in cl(\mathscr{M} + \mathscr{N})$ as $\sum (\frac{1}{n})^2 < \infty$. On the other hand, if $\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1}$ were in $\mathscr{M} + \mathscr{N}$, write

$$\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1} = \sum_{n=1}^{\infty} a_n e_{2n} + \sum_{n=1}^{\infty} b_n (\frac{1}{n} e_{2n-1} + e_{2n}).$$

Since $\{e_n\}$ is an orthonormal basis, we must have $b_n = 1$ and $a_n = -1$ for all $n = 1, 2, \dots$. But then $\sum_{n=1}^{\infty} a_n e_{2n}$ is not in $\mathscr{M}$ (or l_2). Hence, $\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1}$ is not in $\mathscr{M} + \mathscr{N}$.

3. Let $k(x, y) = \sum_{n=0}^{\infty} a_n \cos n(x - y) + b_n \sin n(x - y)$, where a_n, b_n are real numbers with $\sum_n |a_n|^2 + |b_n|^2 < \infty$. Define a linear operator $K \colon \mathscr{L}^2([-\pi, \pi]) \longrightarrow \mathscr{L}^2([-\pi, \pi])$ as

$$(Kf)(x) = \int_{-\pi}^{\pi} f(y) k(x, y)\, dy.$$

Find all the eigenvalues of K.

Solution: Recall that $\{\frac{1}{\sqrt{2\pi}}\} \cup \{\frac{1}{\sqrt{\pi}} \cos nx,\ \frac{1}{\sqrt{\pi}} \sin nx : n = 1, 2, \dots\}$ is an orthonormal basis for $\mathscr{L}^2([-\pi, \pi])$ with the usual inner product defined as

$$< f, g >= \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

Observe that

$$\begin{aligned} k(x, y) &= \sum_{n=0}^{\infty} a_n \cos n(x - y) + b_n \sin n(x - y) \\ &= a_0 + \sum_{n=0}^{\infty} (a_n \cos nx \cos ny + a_n \sin nx \sin ny + b_n \sin nx \cos ny - b_n \cos nx \sin ny). \end{aligned}$$

Every vector $f \in \mathscr{L}^2([-\pi, \pi])$ can be written as

$$f(x) = \frac{\alpha_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{\alpha_n}{\sqrt{\pi}} \cos nx + \sum_{n=1}^{\infty} \frac{\beta_n}{\sqrt{\pi}} \sin nx$$

with $\sum_{n=0}^{\infty} |\alpha_n|^2 + |\beta_n|^2 < \infty$, and for such an f,

$$(Kf)(x) = \int_{-\pi}^{\pi} \left(\frac{\alpha_0}{\sqrt{2\pi}} + \sum_{n=0}^{\infty} \frac{\alpha_n}{\sqrt{\pi}} \cos ny + \sum_{n=0}^{\infty} \frac{\beta_n}{\sqrt{\pi}} \sin ny \right) \cdot$$

$$\left(a_0 + \sum_{n=0}^{\infty} (a_n \cos nx \cos ny + a_n \sin nx \sin ny + b_n \sin nx \cos ny - b_n \cos nx \sin ny) \right) dy$$

$$= a_0 \alpha_0 \sqrt{2\pi} + \sum_{n=1}^{\infty} \sqrt{\pi}(a_n \alpha_n - b_n \beta_n) \cos nx + \sqrt{\pi}(a_n \beta_n + b_n \alpha_n) \sin nx$$

If $Kf = \lambda f$ for some non-zero f, by comparing the coefficients of f and Kf, we have eigenvalue $\lambda_0 = 2\pi a_0$ (corresponds to $f(x) = 1$), and

$$(\pi a_n - \lambda)\alpha_n - \pi b_n \beta_n = 0, \qquad \pi b_n \alpha_n + (\pi a_n - \lambda)\beta_n = 0.$$

Such a system has non-trivial solution if and only if

$$\begin{vmatrix} \pi a_n - \lambda & -\pi b_n \\ \pi b_n & \pi a_n - \lambda \end{vmatrix} = 0.$$

This is equivalent to $(\pi a_n - \lambda)^2 + \pi^2 b_n^2 = 0$, or $\lambda = \pi a_n \pm i\pi b_n$. Therefore $\lambda_n = \pi a_n \pm i\pi b_n$ are eigenvalues (correspond to eigenvectors $f_n(x) = \cos nx \mp i \sin nx$). Hence, the eigenvalues of K are $\lambda_0 = 2\pi a_0$ and $\lambda_n = \pi(a_n \pm i b_n)$ for $n = 1, 2, \ldots$.

4. Let ϕ be a monotonically increasing smooth (continuously differentiable) funtion on $[a, b]$, and ψ be the inverse of ϕ on $[\phi(a), \phi(b)]$. Show that

$$\int_a^b \phi(x)\, dx = \int_{\phi(a)}^{\phi(b)} y\psi'(y)\, dy.$$

Solution: Let μ denote the Lebesgue measure on $[a, b]$. Let

$$P := \{\phi(a) = t_0 < t_1 < t_2 < \cdots < t_n = \phi(b)\}$$

be a partition of $[\phi(a), \phi(b)]$. Since ϕ is monotonically increasing, we have

$$\mu(\{x \in [a, b] \colon t_k \le \phi(x) \le t_{k+1}\}) = \psi(t_{k+1}) - \psi(t_k) = \psi'(\xi_k)(t_{k+1} - t_k)$$

for each k, for some $t_k < \xi_k < t_{k+1}$, where the last equality follows from the Mean Value Theorem applied to ψ. Now for partitions

$$P^{(m)} := \{\phi(a) = t_0^{(m)} < t_1^{(m)} < t_2^{(m)} < \cdots < t_{n_m}^{(m)} = \phi(b)\}$$

such that $\max\{t_{k+1}^{(m)} - t_k^{(m)} \colon k = 1, \dots, n_m\} \longrightarrow 0$ as $m \to \infty$, then

$$\begin{aligned} \int_a^b \phi(x)\,dx &= \lim_{m\to\infty} \sum_k t_k^{(m)} \mu(\{x \in [a,b] \colon t_k^{(m)} \le \phi(x) \le t_{k+1}^{(m)}\}) \\ &= \lim_{m\to\infty} \sum_k t_k^{(m)} \psi'(\xi_k^{(m)})(t_{k+1}^{(m)} - t_k^{(m)}) \\ &= \int_{\phi(a)}^{\phi(b)} y\psi'(y)\,dy. \end{aligned}$$

5. With $(X, \mathcal{M}, \mu)$ some measure space, consider $f \colon X \longrightarrow \bar{\mathbb{R}}$ such that $f \in \mathscr{L}^1(X)$.
   (a) Prove that $\mu(A) = 0$ where $A := \{x \in X \colon |f(x)| = \infty\}$.
   (b) Prove that the set $B := \{x \in X \colon f(x) \neq 0\}$ is σ-finite.
   (c) Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $E \in \mathcal{M}$ with $\mu(E) < \delta$, then $\int_E |f|\,d\mu < \varepsilon$.

Solution:

(a) Follows from Cheyshev's inequality: for all $\alpha > 0$ we have

$$\alpha\mu(\{x \in X \colon |f(x)| \geqslant \alpha\}) \leqslant \int_{\{x\in X \colon |f(x)|\geqslant\alpha\}} |f(x)|\,d\mu \leqslant \|f\|_{\mathscr{L}^1(X)} < \infty.$$

Sending $\alpha \longrightarrow \infty$, the result follows.

(b) Again by Chebyshesh's inequality, we have that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, where

$$E_n := \{x \in X \colon |f(x)| \geqslant 1/n\}.$$

Then we use that $\{x \in X \colon f(x) \neq 0\} = \bigcup_{n=1}^{\infty} E_n$.

(c) For any $\varepsilon > 0$ there exists a simple function $s = \sum_{i=1}^N a_i \mathbf{1}_{E_i}$ with $a_i \in \mathbb{R}$ and $E_i \in \mathcal{M}$ auch that $\int_X |f - s|\,d\mu < \varepsilon/2$. Without loss of generality, we may assume that the E_i are pairwise disjoint. Let $\delta := \varepsilon/(2\max_i |a_i|) > 0$. Then for all $E \in \mathcal{M}$ with $\mu(E) < \delta$ we can estimate

$$\int_E |f|\,d\mu \leqslant \int_X |f - s|\,d\mu + \int_E |s|\,d\mu \leqslant \frac{\varepsilon}{2} + (\max_i |a_i|)\mu(E) < \varepsilon.$$

6. Consider a σ-finite measure space $(X, \mathcal{M}, \mu)$. Fix a function $K \in \mathscr{L}^2(X \times X)$ (defined with respect to the product σ-algebra and product measure) and define a linear map

$$T\colon \mathscr{L}^2(X) \longrightarrow \mathscr{L}^2(X) \quad \text{by} \quad (Tf)(x) := \int_X K(x,y) f(y)\, d\mu(y). \tag{1}$$

Prove that the operator T is well-defined: the integral in (1) converges for μ-almost every $x \in X$ and defines a function in $\mathscr{L}^2(X)$ for all $f \in \mathscr{L}^2(X)$. Moreover, the operator T is bounded with operator norm bounded by $\|T\| \leqslant \|K\|_{\mathscr{L}^2(X\times X)}$.

Solution: For any $f, g \in \mathscr{L}^2(X)$, the map $(x,y) \mapsto g(x)f(y)$ is in $\mathscr{L}^2(X \times X)$ since

$$\int_{X\times X} |g(x)f(y)|^2 \, d\mu \otimes \mu(x,y) = \left(\int_X |g(x)|^2 \, d\mu(x)\right)\left(\int_X |f(x)|^2 \, d\mu(y)\right),$$

which is finite. The Cauchy inequality therefore implies the estimate

$$\left| \int_{X\times X} K(x,y) g(x) f(y) \, d\mu \otimes \mu(x,y) \right| \leqslant \|K\|_{\mathscr{L}^2(X\times X)} \|g\|_{\mathscr{L}^2(X)} \|f\|_{\mathscr{L}^2(X)}.$$

This implies that for any $f \in \mathscr{L}^2(X)$, the map

$$g \mapsto \int_{X\times X} K(x,y) g(x) f(y) \, d\mu \otimes \mu(x,y)$$

is a linear functional on $\mathscr{L}^2(X)$, with norm bounded by $\|K\|_{\mathscr{L}^2(X\times X)} \|f\|_{\mathscr{L}^2(X)}$. By Riesz representation, any such linear functional can be represented by integration against a square integrable function, which is uniquely determined. Since the map $(x,y) \mapsto K(x,y)g(x)f(y)$ is in $\mathscr{L}^1(X \times X)$, by Fubini's theorem we have that

$$\int_{X\times X} K(x,y) g(x) f(y) \, d\mu \otimes \mu(x,y) = \int_X g(x) \int_X K(x,y) f(y) \, d\mu(y) \, d\mu(x)$$

for all f, g. Therefore $\int_{X\times X} K(\cdot, y) f(y) \, d\mu(y) =: Tf$ is a function in $\mathscr{L}^2(X)$ and as such finite μ-a.e. Finally, we can estimate

$$\|Tf\|_{\mathscr{L}^2(X)} = \sup\left\{ \left| \int_X g(x) Tf(x) \, d\mu(x) \right| : \|g\|_{L^2(X)} \leqslant 1 \right\} \leqslant \|K\|_{\mathscr{L}^2(X\times X)} \|f\|_{\mathscr{L}^2(X)},$$

which implies that the operator norm $\|T\| \leqslant \|K\|_{\mathscr{L}^2(X\times X)}$.

7. (a) Assume that μ is a finite measure on a measurable space $(X, \mathcal{M})$. With $q \in (1, \infty)$, let $\{f_k\}_{k=1}^{\infty} \subset \mathscr{L}^q(X)$ and $f \in \mathscr{L}^q(X)$ be given. Suppose that also

   - $\sup_{k\in\mathbb{N}} \|f_k\|_{L^q(X)} < \infty$ and

- $f_k(x) \longrightarrow f(x)$ for μ-a.e. $x \in X$.

Prove that $f_k \longrightarrow f$ in $\mathscr{L}^p(X)$ for all $p \in [1, q)$.

(b) Is the statement in part (a) still true if μ is only assumed to be σ-finite? Justify your answer.

Solution:

(a) By Fatou's lemma, we have that $\int_X |f|^q \, d\mu \leqslant \liminf_{k\to\infty} \int_X |f_k|^q < \infty$. We may therefore consider a sequence $g_k := f_k - f$, which is uniformly bounded in $\mathscr{L}^q(X)$ and converges to zero pointwise almost everywhere. We want to show that $g_k \longrightarrow 0$ strongly in $\mathscr{L}^p(X)$ for all $p \in [1, q)$.

Notice first that by Hölder's and Chebyshev's inequalities, we can estimate

$$\int_X \mathbf{1}_{\{|g_k|>m\}} |g_k|^p \, d\mu \leqslant \|g_k\|^p_{\mathscr{L}^q(X)} \Big(\mu(\{|g_k| > m\})\Big)^{(q-p)/q} \leqslant m^{p-q} \|g_k\|^q_{\mathscr{L}^q(X)} \quad (2)$$

for all $m \geqslant 0$. Let $\tilde{g}_k$ be the function that is equal to g_k if $|g_k| \leqslant m$ and zero otherwise. Then $\tilde{g}_k$ is pointwise bounded by m and the sequence converges to zero almost everywhere. Hence $\tilde{g}_k \longrightarrow 0$ strongly in $\mathscr{L}^p(X)$ by dominated convergence, using the constant function as a majorant and the fact the μ is finite. Combining this with the estimate (2), we obtain the result.

(b) No. If μ is not finite, then a function in $\mathscr{L}^q(X)$ does not even have to be in $\mathscr{L}^p(X)$ for any $p \in [1, q)$. Consider, for (counter)example, the case when $X := [1, \infty)$ equipped with Lebesgue measure. Then $f(x) := 1/(x \log x)^{1/q}$ is in $\mathscr{L}^q(X)$, but not in $\mathscr{L}^p(X)$ for $p < q$ since it does not decay fast enough.

8. Assume that μ is a finite measure on a measurable space $(X, \mathcal{M})$. Let $f\colon X \longrightarrow [0, \infty)$ be $\mathcal{M}$-measurable and $g\colon [0, \infty) \longrightarrow [0, \infty)$ be smooth and increasing. Prove that

$$\int_X g \circ f \, d\mu \geqslant \int_0^\infty g'(t)\mu(\{x \in X \colon f(x) > t\}) \, dt.$$

You may assume without proof that the composition $g \circ f$ is $\mathcal{M}$-measurable.

Solution: Note first that if s is a simple function, then $g \circ s$ is a simple function as well. Moreover, by definition of integrals over nonnegative functions, we have that $\int_X g \circ f \, d\mu = \sup_s \int_X g \circ s \, d\mu$, where the sup is taking over all simple functions s with $0 \leqslant s(x) \leqslant f(x)$ for a.e. $x \in X$. We may therefore assume that f is simple, thus bounded. Let $M := \sup_{x\in X} |f(x)|$, which is finite. We have that

$$g(f(x)) = \int_0^{f(x)} g'(t) \, dt + g(0) \geqslant \int_0^\infty g'(t) \mathbf{1}_{\{f(x)>t\}} \, dt,$$

because $g(0) \geqslant 0$. Since

$$\int_X \int_0^\infty g'(t)\mathbf{1}_{\{f(x)>t\}}\, dt\, d\mu \leqslant g(M)\mu(X) < \infty$$

we can use Fubini's theorem to interchange the order of integration to obtain

$$\begin{aligned}\int_X \int_0^\infty g'(t)\mathbf{1}_{\{f(x)>t\}}\, dt\, d\mu(x) &= \int_0^\infty \int_X g'(t)\mathbf{1}_{\{f(x)>t\}}\, d\mu(x)\, dt \\ &= \int_0^\infty g'(t)\mu(\{f(x) > t\})\, dt.\end{aligned}$$