Analysis Comprehensive Exam Questions
Spring 2013

1. Given $1 < p < \infty$ and $x_n, y \in \ell^p$, show that $x_n \xrightarrow{w} y$ in ℓ^p (weak convergence) if and only if $x_n(k) \to y(k)$ for each k and $\sup \|x_n\|_p < \infty$. Does either implication remain valid if $p = 1$?

Solution: Fix $1 < p < \infty$ and $x_n, y \in \ell^p$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ denote the sequence of standard basis vectors.

$\Rightarrow$. Suppose that $x_n \xrightarrow{w} y$. Then since $\delta_k \in \ell^{p'}$,

$$y(k) \;=\; \langle y, \delta_k\rangle \;=\; \lim_{n\to\infty} \langle x_n, \delta_k\rangle \;=\; \lim_{n\to\infty} x_n(k).$$

This is, x_n converges componentwise to y. All weakly convergent sequences are bounded, so we also have $\sup \|x_n\|_p < \infty$.

$\Leftarrow$. Suppose first that x_n converges componentwise to the zero vector, and that $K = \sup \|x_n\|_p < \infty$. Choose $z \in \ell^{p'}$ and fix $\varepsilon > 0$. Since $p' < \infty$, there exists an $N > 0$ such that $\|z - z_N\|_{p'} < \varepsilon$, where $z_N = \sum_{k=1}^N z(k)\delta_k$. Then

$$\begin{aligned}
\limsup_{n\to\infty} |\langle x_n, z\rangle| &\le \limsup_{n\to\infty} \big(|\langle x_n, z - z_N\rangle| + |\langle x_n, z_N\rangle|\big) \\
&\le \limsup_{n\to\infty} \|x_n\|_p \, \|z - z_N\|_{p'} \;+\; \limsup_{n\to\infty} |\langle x_n, z_N\rangle| \\
&\le K\varepsilon \;+\; \limsup_{n\to\infty} \sum_{k=1}^N |x_n(k)\, z(k)| \\
&\le K\varepsilon \;+\; \sum_{k=1}^N \limsup_{n\to\infty} |x_n(k)\, z(k)| \\
&= K\varepsilon + 0.
\end{aligned}$$

Since ε is arbitrary, we conclude that $\langle x_n, z\rangle \to 0$. Thus $x_n \xrightarrow{w} 0$. The general case follows by replacing x_n with $x_n - y$.

Case $p = 1$. If $p = 1$ then the "$\Rightarrow$" argument remains valid, i.e., if $x_n \xrightarrow{w} y$ in ℓ^1 then x_n converges componentwise to y and $\sup \|x_n\| < \infty$.

However, the converse fails. Set

$$x_n \;=\; \frac{1}{n} \sum_{k=1}^n \delta_k \;=\; \Big(\frac{1}{n}, \dots, \frac{1}{n}, 0, 0, \dots\Big).$$

Then $\|x_n\|_1 = 1$ for all n and x_n converges componentwise to 0. However, x_n does not converge weakly to 0, for if we take $z = (1, 1, 1, \dots) \in \ell^\infty$ then $\langle x_n, z\rangle = 1 \not\to \langle 0, z\rangle$.

2. Suppose that f is a bounded measurable function on a measure space (X, μ). Assume that there exist constants C and $0 < \alpha < 1$ such that

$$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) \leq \frac{C}{\lambda^\alpha}$$

for all $\lambda > 0$. Show that $f \in L^1(X;\mu)$.

Solution: For each $n \in \mathbb{N}$ set

$$X_n := \left\{x \in X : \|f\|_{L^\infty} 2^{-n} \geq |f(x)| > \|f\|_{L^\infty} 2^{-n-1}\right\}.$$

Then $X = \cup_n X_n$ and the X_n are disjoint. Thus, we have that

$$\begin{aligned}
\int_X |f(x)|\, d\mu(x) &= \sum_{n=0}^{\infty} \int_{X_n} |f(x)|\, d\mu(x) \\
&\leq \sum_{n=0}^{\infty} \|f\|_{L^\infty} 2^{-n} \mu\left(X_n\right) \\
&\leq C \|f\|_{L^\infty} \sum_{n=0}^{\infty} 2^{(n+1)\alpha} \|f\|_{L^\infty}^{-\alpha} 2^{-n} \\
&= 2^\alpha C \|f\|_{L^\infty}^{1-\alpha} \sum_{n=0}^{\infty} 2^{(\alpha-1)n} \\
&= C(\alpha) C \|f\|_{L^\infty}^{1-\alpha}.
\end{aligned}$$

In the above estimates, the first inequality follows since the absolute value of f is controlled on X_n, the second follows from the assumption about the measure of μ, and the last equality holds since $0 < \alpha < 1$ and so the series converges.

3. Let (X, μ) be a measure space with $\mu(X) < \infty$ and let $\{f_n\} \in L^1(X; \mu)$ converge to a measurable function f at almost every $x \in X$. Assume there exists a constant C and $p > 1$ such that

$$\sup_{n \geq 1} \int_X |f_n(x)|^p \, d\mu(x) \leq C^p < \infty.$$

Prove

(a) $f \in L^1(X; \mu)$;

(b) $\|f_n - f\|_{L^1(\mu)} \to 0$ as $n \to \infty$.

Solution: (a) By Hölder's Inequality we have that

$$\int_X |f_n(x)| \, d\mu(x) \leq \left(\int_X |f_n(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \mu(X)^{\frac{1}{q}} \leq C\mu(X)^{\frac{1}{q}} < \infty$$

since $\mu(X) < \infty$. Thus, we have that

$$\sup_{n \geq 1} \int_X |f_n(x)| \, d\mu(x) \leq C\mu(X)^{\frac{1}{q}} < \infty.$$

Now, apply Fatou's Theorem to see that

$$\begin{aligned} \int_X |f(x)| \, d\mu(x) &= \int_X \liminf_{n \to \infty} |f_n(x)| \, d\mu(x) \\ &\leq \liminf_{n \to \infty} \int_X |f_n(x)| \, d\mu(x) \\ &\leq \sup_{n \geq 1} \int_X |f_n(x)| \, d\mu(x) < \infty. \end{aligned}$$

So $f \in L^1(X; \mu)$ as claimed.

(b) By Egorov's Theorem, given $\epsilon > 0$ there exists a measurable set $E \subset X$ with $\mu(X \setminus E) < \frac{\epsilon}{4C\mu(X)^{1+\frac{1}{q}}}$ such that $f_k \to f$ uniformly on E. Since $f_n \to f$ uniformly on E there exists an integer N such that for all $n \geq N$ we have that

$$\int_E |f_n(x) - f(x)| \, d\mu(x) < \frac{\epsilon}{2}.$$

Then, for $n \geq N$ we have that

$$\begin{aligned} \int_X |f_n(x) - f(x)| \, d\mu(x) &= \int_{X \setminus E} |f_n(x) - f(x)| \, d\mu(x) + \int_E |f_n(x) - f(x)| \, d\mu(x) \\ &\leq 2C\mu(X)^{\frac{1}{q}+1} \frac{\epsilon}{4C\mu(X)^{1+\frac{1}{q}}} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So we have that $\|f_n - f\|_{L^1(\mu)} \to 0$ as claimed.

4. Let X and Y be Banach spaces and let $T : X \to Y$ be bounded and linear. Show that there is a constant $c > 0$ such that $\|Tx\|_Y \geq c\,\|x\|_X$ for all $x \in X$ if and only if $\ker T = \{0\}$ and $\operatorname{ran} T$ is closed.

Solution: Suppose that $\|Tx\|_Y \geq c\,\|x\|_X$ for all $x \in X$. If we have $x \in \ker T$ then

$$0 = \|Tx\|_Y \geq c\,\|x\|_X\,,$$

which gives $x = 0$, and so $\ker T = \{0\}$. Suppose that $y \in \overline{\operatorname{ran} T}$, and let $y_n \in \operatorname{ran} T$ be such that $y_n \to y$. Since $y_n \in \operatorname{ran} T$, we have that $y_n = Tx_n$ for some $x_n \in X$. Note that $\{x_n\}$ is a Cauchy sequence since

$$\|y_n - y_m\|_Y = \|Tx_n - Tx_m\|_Y \geq c\,\|x_n - x_m\|_X\,.$$

Since X is complete, we have that $x_n \to x$ for some $x \in X$. As T is bounded, hence continuous, we have $Tx_n \to Tx$, and therefore $Tx = y$. Thus $\overline{\operatorname{ran} T} \subset \operatorname{ran} T$, so $\operatorname{ran} T$ is closed.

Now suppose that $\ker T = \{0\}$ and $\operatorname{ran} T$ is closed. Note that since T is bounded and linear and $\ker T = \{0\}$, we have that T is injective. Also, since $Z = \operatorname{ran} T$ is closed, T is a surjective map of X onto the Banach space Z. So, $T\colon X \to Z$ is bounded, linear, and bijective, and so by the Open Mapping Theorem (Bounded Inverse Theorem) we have that $T^{-1}\colon Z \to X$ is bounded. Therefore there is a c such that

$$\left\|T^{-1}y\right\|_X \leq c\,\|y\|_Y\,, \qquad y \in Z = \operatorname{ran} T.$$

Applying this inequality to $y = Tx$, we get the desired result.

5. Let $\{h_n\}_{n\geq 1}$ be a sequence of vectors in a Hilbert space H with the property that $(h_n - h_m) \perp h_m$ whenever $n \geq m$. Then $\sum_n \frac{h_n}{\|h_n\|_H^2}$ converges in H if and only if $\sum_{n\geq 1} \frac{n}{\|h_n\|_H^2} < \infty$.

Solution: We have that $\langle h_n, h_m \rangle_H = \|h_m\|^2$ for all $n \geq m$. Thus

$$\left\| \sum_{k=m}^{n} \frac{h_k}{\|h_k\|_H^2} \right\|_H^2 = \sum_{k=m}^{n} \sum_{l=m}^{n} \frac{\left\|h_{\min\{k,l\}}\right\|_H^2}{\|h_k\|_H^2 \|h_l\|_H^2} = \sum_{k=m}^{n} \frac{2k - 2m + 1}{\|h_k\|_H^2} \tag{1}$$

First, suppose that $\sum_{n\geq 1} \frac{n}{\|h_n\|_H^2} < \infty$. Then by (1) the partial sums of

$$\sum_n \frac{h_n}{\|h_n\|_H^2}$$

form a Cauchy sequence in H, and therefore must converge in H. Conversely, if $\sum_k \frac{h_k}{\|h_k\|_H^2}$ converges in H, then its partial sums are bounded in norm. Using (1) with $m = 1$ show that

$$\sum_{k=1}^{n} \frac{k}{\|h_k\|_H^2} \leq \sum_{k=1}^{n} \frac{2k - 1}{\|h_k\|_H^2} = \left\| \sum_{k=1}^{n} \frac{h_k}{\|h_k\|_H^2} \right\|_H^2 < \infty$$

so $\sum_{n\geq 1} \frac{n}{\|h_n\|_H^2} < \infty$ as claimed.

6. Let $|E|_e$ denote the exterior Lebesgue measure of a set $E \subset \mathbb{R}^n$, and let us define the inner Lebesgue measure of E to be

$$|E|_i = \sup\{|F|_e : F \text{ is closed and } F \subset E\}.$$

(a) Show that if $|E|_e < \infty$, then E is Lebesgue measurable if and only if $|E|_e = |E|_i$.
(b) Is the statement true if $|E|_e = \infty$?

Solution: (a) Let $|A|$ denote the Lebesgue measure of a measurable set A. If E is measurable then for every $\epsilon > 0$ there exists a closed set $F \subset E$ such that $|E \setminus F| < \epsilon$ and therefore $|E| = |E \setminus F| + |F| < \epsilon + |F|$, or equivalently $|F| > |E| - \epsilon$. Since $\epsilon > 0$ is arbitrary we see that $|E|_i \geq |E| = |E|_e$.

Conversely, if $|E|_i = |E|_e$ then there exist an F_σ-set F and a G_δ-set U, such that $F \subset E \subset U$ and $|F| = |E|_i = |E|_e = |U|$. Since $|E|_e < \infty$, we have

$$|U \setminus E|_e \leq |U \setminus F|_e = |U| - |F| = 0,$$

hence E is measurable.

(b) The statement is not true in general if $|E|_e = \infty$. For instance, if N is a non-measurable subset of $[0, 1]^n$ and if we set $E = \mathbb{R}^n \setminus N$, then $|E|_i = |E|_e = \infty$, but E not measurable.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. A collection of functions $\{f_\alpha\}_{\alpha \in A} \subset L^1(\mu)$ is called uniformly integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f_\alpha d\mu| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.
(a) Show that any finite subset of $L^1(\mu)$ is uniformly integrable.
(b) If $\{f_n\}$ is a sequence in $L^1(\mu)$ that converges in the L^1 metric to $f \in L^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

Solution: Note that if $f \in L^1(\mu)$ and $d\nu = f d\mu$, then $\nu \ll \mu$, $d|\nu| = |f|d\mu$ and $|\nu|(X) = \int |f| d\mu < \infty$, i.e. ν is finite. Therefore, the condition $\nu \ll \mu$ can be rewritten in $\epsilon - \delta$ terms as follows: for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \quad \text{implies} \quad |\nu(E)| = \left| \int_E f d\mu \right| < \epsilon. \tag{$**$}$$

Take arbitrary $\epsilon > 0$.
(a) For a finite set $\{f_\alpha\}_{\alpha \in A}$, we can pick $\delta_\alpha > 0$ for the function f_α such that $(**)$ holds, and then take $\delta = \min\{\delta_\alpha : \alpha \in A\}$.
(b) If $f_n \to f$ in L^1, then for every $E \in \mathcal{M}$

$$\left| \int_E f_n d\mu - \int_E f d\mu \right| \leq ||f_n - f||_1 < \frac{\epsilon}{2} \quad \text{for} \quad n \geq N_\epsilon,$$

and therefore

$$\left| \int_E f_n d\mu \right| \leq \left| \int_E f d\mu \right| + \frac{\epsilon}{2} \quad \text{for} \quad n \geq N_\epsilon.$$

The proof now follows similarly to (a) by applying $(**)$ with $\frac{\epsilon}{2}$ for the functions $\{f_1, f_2 \ldots, f_{N_\epsilon - 1}, f\}$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f_n, f, g_n, g for $n \in \mathbb{N}$ be measurable complex-valued functions on X such that $f_n \to f$ in measure and $g_n \to g$ in measure.
(a) Show that $f_n + g_n \to f + g$ in measure.
(b) Show that $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Solution: (a) By the triangle inequality $|(f_n(x) + g_n(x)) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$, we see that

$$\begin{aligned} E_n^\alpha :&= \{x : |(f_n(x) + g_n(x)) - (f(x) + g(x))| \geq \alpha\} \\ &\subset \underbrace{\{x : |f_n(x) - f(x)| \geq \alpha/2\}}_{A_n^\alpha} \cup \underbrace{\{x : |g_n(x) - g(x)| \geq \alpha/2\}}_{B_n^\alpha}. \end{aligned}$$

Thus

$$\mu(E_n^\alpha) \leq \mu(A_n^\alpha) + \mu(B_n^\alpha),$$

and since $\mu(A_n^\alpha) \to 0$, $\mu(B_n^\alpha) \to 0$, as $n \to \infty$, we see that $\mu(E_n^\alpha) \to 0$.

(b) Let $\mu(X) < \infty$ and suppose that the statement is not true. Then, for some $\alpha, \epsilon > 0$ there exists a subsequence $\{f_{n_k} g_{n_k}\}$ of $\{f_n g_n\}$ such that

$$\mu(\{x : |f_{n_k}(x) g_{n_k}(x) - f(x)g(x)| \geq \alpha\}) \geq \epsilon, \quad \text{for all} \quad k \in \mathbb{N}. \tag{2}$$

Since $f_{n_k} \to f$, $g_{n_k} \to g$ in measure, we can find subsequences $\{f_{n_{k_j}}\}$ and $\{g_{n_{k_j}}\}$ such that $f_{n_{k_j}} \to f$ a.e. and $g_{n_{k_j}} \to g$ a.e. Then, $f_{n_{k_j}} g_{n_{k_j}} \to fg$ a.e. and therefore, by Egoroff's theorem, $f_{n_{k_j}} g_{n_{k_j}} \to fg$ almost uniformly, which implies convergence in measure, contradicting (2).

As a simple counterexample when $\mu(X) = \infty$, consider $\mathbb{R}$ with the Lebesgue measure. Then $f_n(x) = x + \frac{1}{n} \to f(x) = x$ in measure, but $f_n^2(x) \not\to f^2(x)$ in measure.