1. Given $1 < p < \infty$ and $x_n, y \in \ell^p$, show that $x_n \overset{w}{\rightharpoonup} y$ in $\ell^p$ (weak convergence) if and only if $x_n(k) \to y(k)$ for each $k$ and $\sup \|x_n\|_p < \infty$. Does either implication remain valid if $p = 1$?

**Solution:** Fix $1 < p < \infty$ and $x_n, y \in \ell^p$. Let $\{\delta_n\}_{n \in \mathbb{N}}$ denote the sequence of standard basis vectors.

$\Rightarrow$. Suppose that $x_n \overset{w}{\rightharpoonup} y$. Then since $\delta_k \in \ell^{p'}$, $y(k) = \langle y, \delta_k \rangle = \lim_{n \to \infty} \langle x_n, \delta_k \rangle = \lim_{n \to \infty} x_n(k)$.

This is, $x_n$ converges componentwise to $y$. All weakly convergent sequences are bounded, so we also have $\sup \|x_n\|_p < \infty$.

$\Leftarrow$. Suppose first that $x_n$ converges componentwise to the zero vector, and that $K = \sup \|x_n\|_p < \infty$. Choose $z \in \ell^{p'}$ and fix $\varepsilon > 0$. Since $p' < \infty$, there exists an $N > 0$ such that $\|z - z_N\|_{p'} < \varepsilon$, where $z_N = \sum_{k=1}^{N} z(k) \delta_k$. Then

$$
\limsup_{n \to \infty} \|\langle x_n, z \rangle\| \leq \limsup_{n \to \infty} \left( |\langle x_n, z - z_N \rangle| + |\langle x_n, z_N \rangle| \right) \\
\leq \limsup_{n \to \infty} \|x_n\|_p \|z - z_N\|_{p'} + \limsup_{n \to \infty} |\langle x_n, z_N \rangle| \\
\leq K \varepsilon + \limsup_{n \to \infty} \sum_{k=1}^{N} |x_n(k) z(k)| \\
\leq K \varepsilon + \sum_{k=1}^{N} \limsup_{n \to \infty} |x_n(k) z(k)| \\
= K \varepsilon + 0.
$$

Since $\varepsilon$ is arbitrary, we conclude that $\langle x_n, z \rangle \to 0$. Thus $x_n \overset{w}{\rightharpoonup} 0$. The general case follows by replacing $x_n$ with $x_n - y$.

*Case $p = 1$. If $p = 1$ then the “$\Rightarrow$” argument remains valid, i.e., if $x_n \overset{w}{\rightharpoonup} y$ in $\ell^1$ then $x_n$ converges componentwise to $y$ and $\sup \|x_n\| < \infty$.

However, the converse fails. Set

$$
x_n = \frac{1}{n} \sum_{k=1}^{n} \delta_k = \left( \frac{1}{n}, \ldots, \frac{1}{n}, 0, 0, \ldots \right).
$$

Then $\|x_n\|_1 = 1$ for all $n$ and $x_n$ converges componentwise to $0$. However, $x_n$ does not converge weakly to $0$, for if we take $z = (1, 1, 1, \ldots) \in \ell^\infty$ then $\langle x_n, z \rangle = 1 \not\to \langle 0, z \rangle$. 
2. Suppose that $f$ is a bounded measurable function on a measure space $(X, \mu)$. Assume that there exist constants $C$ and $0 < \alpha < 1$ such that

$$
\mu \left( \{ x \in X : |f(x)| > \lambda \} \right) \leq \frac{C}{\lambda^\alpha}
$$

for all $\lambda > 0$. Show that $f \in L^1(X; \mu)$.

**Solution:** For each $n \in \mathbb{N}$ set

$$
X_n := \{ x \in X : \|f\|_{L^\infty} \cdot 2^{-n} \geq |f(x)| > \|f\|_{L^\infty} \cdot 2^{-n-1} \}.
$$

Then $X = \bigcup_n X_n$ and the $X_n$ are disjoint. Thus, we have that

$$
\int_X |f(x)| \, d\mu(x) = \sum_{n=0}^{\infty} \int_{X_n} |f(x)| \, d\mu(x) \\
\leq \sum_{n=0}^{\infty} \|f\|_{L^\infty} \cdot 2^{-n} \mu(X_n) \\
\leq C \|f\|_{L^\infty} \sum_{n=0}^{\infty} 2^{(n+1)\alpha} \|f\|_{L^\infty}^{-\alpha} \cdot 2^{-n} \\
= 2^\alpha C \|f\|_{L^\infty}^{1-\alpha} \sum_{n=0}^{\infty} 2^{(\alpha-1)n} \\
= C(\alpha) C \|f\|_{L^\infty}^{1-\alpha}.
$$

In the above estimates, the first inequality follows since the absolute value of $f$ is controlled on $X_n$, the second follows from the assumption about the measure of $\mu$, and the last equality holds since $0 < \alpha < 1$ and so the series converges.
3. Let \((X, \mu)\) be a measure space with \(\mu(X) < \infty\) and let \(\{f_n\} \in L^1(X; \mu)\) converge to a measurable function \(f\) at almost every \(x \in X\). Assume there exists a constant \(C\) and \(p > 1\) such that
\[
\sup_{n \geq 1} \int_X |f_n(x)|^p \, d\mu(x) \leq C^p < \infty.
\]
Prove
(a) \(f \in L^1(X; \mu)\);
(b) \(\|f_n - f\|_{L^1(\mu)} \to 0\) as \(n \to \infty\).

**Solution:** (a) By Hölder’s Inequality we have that
\[
\int_X |f_n(x)| \, d\mu(x) \leq \left( \int_X |f_n(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \mu(X)^{\frac{1}{q}} \leq C \mu(X)^{\frac{1}{q}} < \infty
\]
since \(\mu(X) < \infty\). Thus, we have that
\[
\sup_{n \geq 1} \int_X |f_n(x)| \, d\mu(x) \leq C \mu(X)^{\frac{1}{q}} < \infty.
\]

Now, apply Fatou’s Theorem to see that
\[
\int_X |f(x)| \, d\mu(x) = \int_X \liminf_{n \to \infty} |f_n(x)| \, d\mu(x)
\leq \liminf_{n \to \infty} \int_X |f_n(x)| \, d\mu(x)
\leq \sup_{n \geq 1} \int_X |f_n(x)| \, d\mu(x) < \infty.
\]
So \(f \in L^1(X; \mu)\) as claimed.

(b) By Egorov’s Theorem, given \(\epsilon > 0\) there exists a measurable set \(E \subset X\) with \(\mu(X \setminus E) < \frac{\epsilon}{4C\mu(X)^{1+\frac{1}{q}}}\) such that \(f_k \to f\) uniformly on \(E\). Since \(f_n \to f\) uniformly on \(E\) there exists an integer \(N\) such that for all \(n \geq N\) we have that
\[
\int_E |f_n(x) - f(x)| \, d\mu(x) < \frac{\epsilon}{2}.
\]
Then, for \(n \geq N\) we have that
\[
\int_X |f_n(x) - f(x)| \, d\mu(x) = \int_{X \setminus E} |f_n(x) - f(x)| \, d\mu(x) + \int_E |f_n(x) - f(x)| \, d\mu(x)
\leq 2C \mu(X)^{\frac{1}{q}+1} \frac{\epsilon}{4C\mu(X)^{1+\frac{1}{q}}} + \frac{\epsilon}{2} = \epsilon.
\]
So we have that \(\|f_n - f\|_{L^1(\mu)} \to 0\) as claimed.
4. Let $X$ and $Y$ be Banach spaces and let $T : X \to Y$ be bounded and linear. Show that there is a constant $c > 0$ such that $\|Tx\|_Y \geq c \|x\|_X$ for all $x \in X$ if and only if $\text{ker} \, T = \{0\}$ and $\text{ran} \, T$ is closed.

Solution: Suppose that $\|Tx\|_Y \geq c \|x\|_X$ for all $x \in X$. If we have $x \in \text{ker} \, T$ then

$$0 = \|Tx\|_Y \geq c \|x\|_X,$$

which gives $x = 0$, and so $\text{ker} \, T = \{0\}$. Suppose that $y \in \overline{\text{ran} \, T}$, and let $y_n \in \text{ran} \, T$ be such that $y_n \to y$. Since $y_n \in \text{ran} \, T$, we have that $y_n = Tx_n$ for some $x_n \in X$. Note that $\{x_n\}$ is a Cauchy sequence since

$$\|y_n - y_m\|_Y = \|Tx_n - Tx_m\|_Y \geq c \|x_n - x_m\|_X.$$

Since $X$ is complete, we have that $x_n \to x$ for some $x \in X$. As $T$ is bounded, hence continuous, we have $Tx_n \to Tx$, and therefore $Tx = y$. Thus $\overline{\text{ran} \, T} \subset \text{ran} \, T$, so $\text{ran} \, T$ is closed.

Now suppose that $\text{ker} \, T = \{0\}$ and $\text{ran} \, T$ is closed. Note that since $T$ is bounded and linear and $\text{ker} \, T = \{0\}$, we have that $T$ is injective. Also, since $Z = \text{ran} \, T$ is closed, $T$ is a surjective map of $X$ onto the Banach space $Z$. So, $T : X \to Z$ is bounded, linear, and bijective, and so by the Open Mapping Theorem (Bounded Inverse Theorem) we have that $T^{-1} : Z \to X$ is bounded. Therefore there is a $c$ such that

$$\|T^{-1}y\|_X \leq c \|y\|_Y, \quad y \in Z = \text{ran} \, T.$$

Applying this inequality to $y = Tx$, we get the desired result.
5. Let \( \{h_n\}_{n \geq 1} \) be a sequence of vectors in a Hilbert space \( H \) with the property that \((h_n - h_m) \perp h_m \) whenever \( n \geq m \). Then \( \sum_n \frac{h_n}{\|h_n\|_H} \) converges in \( H \) if and only if \( \sum_{n \geq 1} \frac{n}{\|h_n\|_H} < \infty \).

**Solution:** We have that \( \langle h_n, h_m \rangle_H = \|h_m\|^2 \) for all \( n \geq m \). Thus

\[
\left\| \sum_{k=m}^{n} \frac{h_k}{\|h_k\|_H} \right\|_H^2 = \sum_{k=m}^{n} \sum_{l=m}^{n} \frac{\|h_{\min\{k,l\}}\|_H^2}{\|h_k\|_H \|h_l\|_H} = \sum_{k=m}^{n} \frac{2k - 2m + 1}{\|h_k\|_H^2}
\]

(1)

First, suppose that \( \sum_{n \geq 1} \frac{n}{\|h_n\|_H} < \infty \). Then by (1) the partial sums of

\[
\sum_n \frac{h_n}{\|h_n\|_H}
\]

form a Cauchy sequence in \( H \), and therefore must converge in \( H \). Conversely, if \( \sum_k \frac{h_k}{\|h_k\|_H} \) converges in \( H \), then its partial sums are bounded in norm. Using (1) with \( m = 1 \) show that

\[
\sum_{k=1}^{n} \frac{k}{\|h_k\|_H^2} \leq \sum_{k=1}^{n} \frac{2k - 1}{\|h_k\|_H^2} = \left\| \sum_{k=1}^{n} \frac{h_k}{\|h_k\|_H^2} \right\|_H^2 < \infty
\]

so \( \sum_{n \geq 1} \frac{n}{\|h_n\|_H} < \infty \) as claimed.
6. Let $|E|_e$ denote the exterior Lebesgue measure of a set $E \subset \mathbb{R}^n$, and let us define the inner Lebesgue measure of $E$ to be

$$|E|_i = \sup\{|F|_e : F \text{ is closed and } F \subset E\}.$$  

(a) Show that if $|E|_e < \infty$, then $E$ is Lebesgue measurable if and only if $|E|_e = |E|_i$.

(b) Is the statement true if $|E|_e = \infty$?

**Solution:** (a) Let $|A|$ denote the Lebesgue measure of a measurable set $A$. If $E$ is measurable then for every $\epsilon > 0$ there exists a closed set $F \subset E$ such that $|E \setminus F| < \epsilon$ and therefore $|E| = |E \setminus F| + |F| < \epsilon + |F|$, or equivalently $|F| > |E| - \epsilon$. Since $\epsilon > 0$ is arbitrary we see that $|E|_i \geq |E| = |E|_e$.

Conversely, if $|E|_i = |E|_e$ then there exist an $F_\sigma$-set $F$ and a $G_\delta$-set $U$, such that $F \subset E \subset U$ and $|F| = |E|_i = |E|_e = |U|$. Since $|E|_e < \infty$, we have

$$|U \setminus E|_e \leq |U \setminus F|_e = |U| - |F| = 0,$$

hence $E$ is measurable.

(b) The statement is not true in general if $|E|_e = \infty$. For instance, if $N$ is a non-measurable subset of $[0,1]^n$ and if we set $E = \mathbb{R}^n \setminus N$, then $|E|_i = |E|_e = \infty$, but $E$ not measurable.
7. Let \((X, \mathcal{M}, \mu)\) be a measure space. A collection of functions \(\{f_\alpha\}_{\alpha \in A} \subset L^1(\mu)\) is called uniformly integrable if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(\left| \int_E f_\alpha d\mu \right| < \epsilon\) for all \(\alpha \in A\) whenever \(\mu(E) < \delta\).

(a) Show that any finite subset of \(L^1(\mu)\) is uniformly integrable.
(b) If \(\{f_n\}\) is a sequence in \(L^1(\mu)\) that converges in the \(L^1\) metric to \(f \in L^1(\mu)\), then \(\{f_n\}\) is uniformly integrable.

**Solution:** Note that if \(f \in L^1(\mu)\) and \(d\nu = fd\mu\), then \(\nu \ll \mu, d|\nu| = |f|d\mu\) and \(|\nu|(X) = \int |f|d\mu < \infty\), i.e. \(\nu\) is finite. Therefore, the condition \(\nu \ll \mu\) can be rewritten in \(\epsilon - \delta\) terms as follows: for every \(\epsilon > 0\) there exists \(\delta > 0\) such that

\[
\mu(E) < \delta \implies |\nu(E)| = \left| \int_E fd\mu \right| < \epsilon. \quad (**)
\]

Take arbitrary \(\epsilon > 0\).
(a) For a finite set \(\{f_\alpha\}_{\alpha \in A}\), we can pick \(\delta_\alpha > 0\) for the function \(f_\alpha\) such that (**) holds, and then take \(\delta = \min\{\delta_\alpha : \alpha \in A\}\).
(b) If \(f_n \to f\) in \(L^1\), then for every \(E \in \mathcal{M}\)

\[
\left| \int_E f_n d\mu - \int_E f d\mu \right| \leq ||f_n - f||_1 < \frac{\epsilon}{2} \quad \text{for} \quad n \geq N_\epsilon,
\]

and therefore

\[
\left| \int_E f_n d\mu \right| \leq \left| \int_E f d\mu \right| + \frac{\epsilon}{2} \quad \text{for} \quad n \geq N_\epsilon.
\]

The proof now follows similarly to (a) by applying (**) with \(\frac{\epsilon}{2}\) for the functions \(\{f_1, f_2, \ldots, f_{N_\epsilon-1}, f\}\).
8. Let \((X, \mathcal{M}, \mu)\) be a measure space, and let \(f_n, f, g, g_n\) for \(n \in \mathbb{N}\) be measurable complex-valued functions on \(X\) such that \(f_n \to f\) in measure and \(g_n \to g\) in measure.

(a) Show that \(f_n + g_n \to f + g\) in measure.

(b) Show that \(f_n g_n \to fg\) in measure if \(\mu(X) < \infty\), but not necessarily if \(\mu(X) = \infty\).

**Solution:**

(a) By the triangle inequality 
\[
|((f_n(x) + g_n(x)) - (f(x) + g(x)))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|,
\]
we see that

\[
E^\alpha_n : = \{x : |(f_n(x) + g_n(x)) - (f(x) + g(x))| \geq \alpha\} \subset A^n_\alpha \cup B^n_\alpha.
\]
Thus

\[
\mu(E^\alpha_n) \leq \mu(A^n_\alpha) + \mu(B^n_\alpha),
\]
and since \(\mu(A^n_\alpha) \to 0, \mu(B^n_\alpha) \to 0\), as \(n \to \infty\), we see that \(\mu(E^\alpha_n) \to 0\).

(b) Let \(\mu(X) < \infty\) and suppose that the statement is not true. Then, for some \(\alpha, \epsilon > 0\) there exists a subsequence \(\{f_{n_k}g_{n_k}\}\) of \(\{f_n g_n\}\) such that

\[
\mu(\{x : |f_{n_k}(x)g_{n_k}(x) - f(x)g(x)| \geq \alpha\}) \geq \epsilon, \quad \text{for all } k \in \mathbb{N}. \quad (2)
\]

Since \(f_{n_k} \to f, g_{n_k} \to g\) in measure, we can find subsequences \(\{f_{n_k_j}\}\) and \(\{g_{n_k_j}\}\) such that \(f_{n_k_j} \to f\) a.e. and \(g_{n_k_j} \to g\) a.e. Then, \(f_{n_k_j}g_{n_k_j} \to fg\) a.e. and therefore, by Egoroff’s theorem, \(f_{n_k_j}g_{n_k_j} \to fg\) almost uniformly, which implies convergence in measure, contradicting (2).

As a simple counterexample when \(\mu(X) = \infty\), consider \(\mathbb{R}\) with the Lebesgue measure. Then \(f_n(x) = x + \frac{1}{n} \to f(x) = x\) in measure, but \(f^2_n(x) \not\to f^2(x)\) in measure.