1. (a) Prove that the polynomial \( f(X) = X^6 + X^3 + 1 = (X^9 - 1)/(X^3 - 1) \) is irreducible over \( \mathbb{Q} \).
   (b) Find the factorization of \( f(X) \) over \( \mathbb{F}_{19} \).

2. Which of the following rings are isomorphic? Give justifications.
   (a) \( R_1 = \mathbb{Z}[i]/(5) \)
   (b) \( R_2 = \mathbb{F}_5[X]/(X^2 - 1) \)
   (c) \( R_3 = \mathbb{F}_5[X]/(X^2 + 1) \)

   Here \( \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \) is the ring of Gauss integers.

3. Let \( K \) be the splitting field over \( \mathbb{Q} \) for an irreducible polynomial of degree 3. What are the possibilities for \([K : \mathbb{Q}]\)? Give an example to show that each possibility does occur.

4. Let \( p, q \) be prime numbers with \( p < q \). Prove that there exists a non-cyclic group of order \( pq \) if and only if \( p \mid q - 1 \).

5. Let \( G \) be a finite group, and let \( H \) be a subgroup of \( G \) of index \( p \), with \( p \) prime. If \( n_H \) denotes the number of subgroups of \( G \) conjugate to \( H \), prove that \( n_H = 1 \) if \( H \) is normal in \( G \), and that \( n_H = p \) otherwise.

6. Let \( I \) be a nonzero ideal in \( \mathbb{Z}[X] \), and suppose that the lowest degree of a nonzero polynomial in \( I \) is \( n \) and that \( I \) contains some monic polynomial of degree \( n \). Prove that \( I \) is a principal ideal.

7. (a) If \( n \) is prime and \( F(X) \) is an irreducible polynomial over \( \mathbb{Q} \) of degree \( n \), prove that the Galois group of \( F \) over \( \mathbb{Q} \) contains an \( n \)-cycle.
(b) If \( n \) is not prime, show that the Galois group in part (a) need not contain an \( n \)-cycle.
[**Hint:** Consider the cyclotomic polynomial \( \Phi_8(X) \).]

8. Let \( P \) be the vector space of all real polynomials and let \( L : P \rightarrow P \) be the linear transformation defined by \( L(f) = f + f' \), where \( f' \) is the derivative of \( f \). Prove that \( L \) is invertible.