Algebra Comprehensive Exam — Spring 2015
January 16, 2015

Student Number:

Instructions: Complete five of the eight problems below, and **circle** their numbers exactly in the box below – the uncircled problems will **not** be graded.

1	2	3	4	5	6	7	8

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. (a) Prove that the polynomial $f(X) = X^6 + X^3 + 1 = (X^9 - 1)/(X^3 - 1)$ is irreducible over $\mathbf{Q}$.
   (b) Find the factorization of $f(X)$ over $\mathbf{F}_{19}$.

2. Which of the following rings are isomorphic? Give justifications.
   (a) $R_1 = \mathbf{Z}[i]/(5)$
   (b) $R_2 = \mathbf{F}_5[X]/(X^2 - 1)$
   (c) $R_3 = \mathbf{F}_5[X]/(X^2 + 1)$

   Here $\mathbf{Z}[i] = \{a + bi \ : \ a, b \in \mathbf{Z}\}$ is the ring of Gauss integers.

3. Let K be the splitting field over $\mathbf{Q}$ for an *irreducible* polynomial of degree 3. What are the possibilities for $[K : \mathbf{Q}]$? Give an example to show that each possibility does occur.

4. Let p, q be prime numbers with $p < q$. Prove that there exists a non-cyclic group of order pq if and only if $p \mid q - 1$.

5. Let G be a finite group, and let H be a subgroup of G of index p, with p prime. If n_H denotes the number of subgroups of G conjugate to H, prove that $n_H = 1$ if H is normal in G, and that $n_H = p$ otherwise.

6. Let I be a nonzero ideal in $\mathbf{Z}[X]$, and suppose that the lowest degree of a nonzero polynomial in I is n and that I contains some monic polynomial of degree n. Prove that I is a principal ideal.

7. (a) If n is prime and $F(X)$ is an irreducible polynomial over $\mathbf{Q}$ of degree n, prove that the Galois group of F over $\mathbf{Q}$ contains an n-cycle.

(b) If n is not prime, show that the Galois group in part (a) need not contain an n-cycle. [**Hint:** Consider the cyclotomic polynomial $\Phi_8(X)$.]

8. Let P be the vector space of all real polynomials and let $L : P \to P$ be the linear transformation defined by $L(f) = f + f'$, where f' is the derivative of f. Prove that L is invertible.