Analysis Comprehensive Exam — Spring 2015
January 9, 2015

Student Number:

Instructions: Complete five of the eight problems below, and **circle** their numbers exactly in the box below – the uncircled problems will **not** be graded.

1	2	3	4	5	6	7	8

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. For $A, B \subset \mathbb{R}^d$ define $A + B = \{a + b : a \in A, b \in B\}$.
(a) Show that if A and B are F_σ sets, then $A + B$ is also an F_σ set.
(b) Give an example of Lebesgue null sets A and B in $\mathbb{R}^2$ for which $A + B$ is not measurable.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and E_i, $i = 1, 2, \ldots$, be measurable subsets of X. Assume that for some $\alpha > 0$ we have

$$\sum_{i=1}^{\infty} \mu(E_i)^\alpha < \infty.$$

For which values of α can you say that

$$\mu\left(\limsup_{n\to\infty} E_n\right) = 0.$$

Here $\limsup_{n\to\infty} E_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$.

3. Let $f \in L^p([0, \infty])$. Show that for $1 < p < \infty$ we have

$$\lim_{x\to\infty} \frac{1}{x^{1-\frac{1}{p}}} \int_0^x f(t)dt = 0.$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, f_1, f_2, \ldots$ be nonnegative measurable functions such that $f_n \to f$ in measure.
(a) Prove that $\int f \le \liminf \int f_n$.
(b) Give an example that shows that the inequality in (a) can be strict.

5. Show that $\int_0^\infty \frac{\sin^2 x}{x} e^{-sx}\, dx = \frac{\ln(1+4/s^2)}{4}$ for $s > 0$ by applying the Fubini's theorem to the function $f(x,y) = e^{-sx}\sin(2xy)$ on $E = (0,\infty) \times (0,1)$.

6. Let H be an Hilbert space and $x_k \in H$ be a sequence such that x_k converges weakly to x. Show that there exists a subsequence x_{k_i} such that

$$\frac{1}{N}\sum_{i=1}^{N} x_{k_i}$$

converges to x in norm.

7. Assume that f has bounded variation on $[a,b]$. Letting $V[f;a,b]$ denote the total variation of f on $[a,b]$, prove that

$$\int_a^b |f'| \leq V[f;a,b].$$

8. Show that $L^2[0,1]$ is a meager (or, equivalently, of the first category) subset of $L^1[0,1]$.