Discrete Mathematics Comprehensive Exam
— Spring 2015 —
January 14, 2015

Student Number:

Instructions: Complete five of the seven problems below, and **circle** their numbers exactly in the box below – the uncircled problems will **not** be graded.

1	2	3	4	5	6	7

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. A grocery store is preparing holiday fruit baskets for sale. Each fruit basket will have n pieces of fruit in it, chosen from apples, pears, oranges, and grapefruit. How many different ways can such a basket be prepared if there must be at least one apple in a basket, a basket cannot contain more than three pears, and the number of oranges must be a multiple of four?

2. Let $T_1, T_2, \ldots, T_k$ be subtrees of a tree T such that any two of them have a vertex in common. Prove that they all have a vertex in common.

3. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of points in the plane such that the distance between any two distinct points in S is at least one. Show that there are at most $3n$ pairs of points at distance exactly one.

4. Prove that for every integer $k \geq 1$ there exists an integer N such that if the subsets of $\{1, 2, \ldots, N\}$ are colored using k colors, then there exist disjoint non-empty sets $X, Y \subseteq \{1, 2, \ldots, N\}$ such that X, Y and $X \cup Y$ receive the same color.
   Hint. You may want to consider intervals.

5. Prove that every three-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $c\sqrt{n^3/m}$, for a suitable constant $c > 0$.

6. Given an $n \times n$ matrix $A = \{a_{ij} = \pm 1\}$ of lights, and row switches x_i and column switches y_j (for $1 \leq i, j \leq n$), the objective is to find $x_i \in \{\pm 1\}$ and $y_j \in \{\pm 1\}$ so as to maximize the discrepancy $|\sum_{i,j} a_{ij} x_i y_j|$.

   Show that there exists an A for which this maximum cannot be made larger (no matter what x's and y's are chosen) than $cn^{3/2}$, for a suitable constant $c > 0$.

   Hint. You may use probabilistic method together with the large deviation result: when $S_n = \sum_{i=1}^{n} X_i$, with X_i i.i.d. random variables taking values in $\{+1, -1\}$ with equal probability, then
   $$\Pr[|S_n| > \lambda] < 2e^{-\lambda^2/2n}.$$

7. A 0-1 matrix is "nine-free" if it has no 3×3 submatrix of all 1's. (Note that the rows and columns in the submatrix need not be consecutive.) Show that there exists a nine-free $n \times n$ 0-1 matrix of at least $cn^{3/2}$ 1's, for a suitable constant $c > 0$.