Numerical Analysis Comprehensive Exam
— Spring 2015 —
January 7, 2015

Student Number:

Instructions: Complete five of the eight problems below, **among the five questions, at least three of them must be selected from the last four problems. Circle** the numbers of the five problems you want graded in the box below – the uncircled problems will **not** be graded.

1	2	3	4	5	6	7	8

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Derive the two-point Gaussian quadrature on $[-h, h]$

$$\int_{-h}^{h} f(x)\, dx = h\{f(-a) + f(a)\} + R,$$

compute the value of a and find an asymptotic estimation of the error R.

2. Find a formula for the following polynomial interpolation problem. Let $x_i = x_0 + ih$, $i = 0, 1, 2, 3$, $h > 0$. Find a polynomial $p(x)$ of degree ≤ 5 for which

$$\begin{aligned} p(x_i) = f(x_i), &\quad i = 0, 1, 2, 3; \\ p'(x_0) = f'(x_0), &\quad p'(x_3) = f'(x_3), \end{aligned} \tag{1}$$

where $f(x)$ has continuous derivative of any order in $\mathbb{R}$. Derive an error formula for $f(x) - p(x)$. What's the order of approximation for $x \in [x_0, x_3]$?

3. The iteration $x_{n+1} = 2 - (1 + c)x_n + cx_n^3$ will converge to $x^* = 1$ for some value of c (provided x_0 is chosen sufficiently close to x^*). Find the values of c for which this is true. For what value of c will the convergence be quadratic?

4. A numerical scheme

$$y_{n+1} = y_n + \frac{h}{2}[3f_n - f_{n-1}]$$

is used to solve the initial value problem

$$\begin{cases} y' = f(y, t) \\ y(0) = y_0 \end{cases}$$

Determine the local truncation error of the numerical scheme. What is the global order of accuracy of the scheme? Is the scheme zero-stable?

5. Consider a matrix

$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix},$$

where ϵ is a small positive number.

(a) What is the condition number of matrix A, measured in $\infty-$norm?

(b) Take $\epsilon = 0.01$. If one tries to solve $A\vec{x} = \vec{b}$ by an backward stable method, and makes a perturbation to $\vec{b}$ by $\delta\vec{b}$, with $\frac{\|\delta\vec{b}\|}{\|\vec{b}\|} \leq 10^{-8}$, what is the relative error that will be expected in the solution?

6. One uses Conjugate Gradient method to solve a linear system of equations with symmetric positive definite matrix A. Assuming the computation leads to the error $\|e_0\|_A = 1$ and $\|e_8\|_A = 2^{-7}$, where $e_n = x_n - x^*$ and x^* is the solution. Based solely on this data,

(a) What bound can you give on the condition number $\kappa(A)$?

(b) What bound can you give on $\|e_{16}\|_A$?

7. Consider the equation $u_t + au_x = \gamma u_{xx}$ where a and $\gamma > 0$ are constants. Study the l_2 (average) and l_∞ stability of the following scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = \gamma\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

where Δt and Δx are the temporal and spatial mesh sizes respectively, U_j^n is supposed to approximate $u(j\Delta x, n\Delta t)$.

8. For the Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$, consider the numerical scheme

$$\frac{U_j^{n+1} - (1/2)(U_{j-1}^n + U_{j+1}^n)}{\Delta t} + U_j^n\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0.$$

Is it a good scheme for the equation? why? Can you correct the scheme? Justify your answer and find the truncation error of the new scheme. If its truncation error has a term proportional to $O(1/\Delta t)$, could you further improve the scheme to remove this term?