Student Number:

Instructions: Complete five of the eight problems below, among the five questions, at least three of them must be selected from the last four problems. Circle the numbers of the five problems you want graded in the box below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write only on the front side of the solution pages. Work on the back of the page might not be graded.

1. Derive the two-point Gaussian quadrature on $[-h, h]$

$$\int_{-h}^{h} f(x) \, dx = h\{f(-a) + f(a)\} + R,$$

compute the value of $a$ and find an asymptotic estimation of the error $R$.

2. Find a formula for the following polynomial interpolation problem. Let $x_i = x_0 + ih$, $i = 0, 1, 2, 3, \ h > 0$. Find a polynomial $p(x)$ of degree $\leq 5$ for which

$$p(x_i) = f(x_i), \quad i = 0, 1, 2, 3;$$

$$p'(x_0) = f'(x_0), \quad p'(x_3) = f'(x_3),$$

(1)

where $f(x)$ has continuous derivative of any order in $\mathbb{R}$. Derive an error formula for $f(x) - p(x)$. What’s the order of approximation for $x \in [x_0, x_3]$?

3. The iteration $x_{n+1} = 2 - (1 + c)x_n + cx_n^3$ will converge to $x^* = 1$ for some value of $c$ (provided $x_0$ is chosen sufficiently close to $x^*$). Find the values of $c$ for which this is true. For what value of $c$ will the convergence be quadratic?

4. A numerical scheme

$$y_{n+1} = y_n + \frac{h}{2}[3f_n - f_{n-1}]$$

is used to solve the initial value problem

$$\begin{cases} y' = f(y, t) \\ y(0) = y_0 \end{cases}$$

Determine the local truncation error of the numerical scheme. What is the global order of accuracy of the scheme? Is the scheme zero-stable?
5. Consider a matrix
\[ A = \begin{bmatrix} 1 & 1 + \epsilon \\ 1 - \epsilon & 1 \end{bmatrix}, \]
where \( \epsilon \) is a small positive number.

(a) What is the condition number of matrix \( A \), measured in \( \infty \)-norm?

(b) Take \( \epsilon = 0.01 \). If one tries to solve \( A\vec{x} = \vec{b} \) by an backward stable method, and makes a perturbation to \( \vec{b} \) by \( \delta \vec{b} \), with \( \|\delta \vec{b}\| \leq 10^{-8} \), what is the relative error that will be expected in the solution?

6. One uses Conjugate Gradient method to solve a linear system of equations with symmetric positive definite matrix \( A \). Assuming the computation leads to the error \( \|e_0\|_A = 1 \) and \( \|e_8\|_A = 2^{-7} \), where \( e_n = x_n - x^* \) and \( x^* \) is the solution. Based solely on this data,

(a) What bound can you give on the condition number \( \kappa(A) \)?

(b) What bound can you give on \( \|e_{16}\|_A \)?

7. Consider the equation \( u_t + au_x = \gamma u_{xx} \) where \( a \) and \( \gamma > 0 \) are constants. Study the \( l_2 \) (average) and \( l_\infty \) stability of the following scheme
\[ \frac{U_{j+1}^n - U_j^n}{\Delta t} + a \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = \gamma \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \]
where \( \Delta t \) and \( \Delta x \) are the temporal and spatial mesh sizes respectively, \( U_j^n \) is supposed to approximate \( u(j\Delta x, n\Delta t) \).

8. For the Burgers’ equation \( u_t + (\frac{1}{2}u^2)_x = 0 \), consider the numerical scheme
\[ \frac{U_{j+1}^n - (1/2)(U_{j-1}^n + U_{j+1}^n)}{\Delta t} + U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0. \]
Is it a good scheme for the equation? why? Can you correct the scheme? Justify your answer and find the truncation error of the new scheme. If its truncation error has a term proportional to \( O(1/\Delta t) \), could you further improve the scheme to remove this term?