Probability Comprehensive Exam — Spring 2014
January 7, 2015

Student Number:

Instructions: Complete five of the eight problems below, and **circle** their numbers exactly in the box below – the uncircled problems will **not** be graded.

1	2	3	4	5	6	7	8

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Assume X is a symmetric random variable such that $\mathbb{E}[X^2] = 1$ and $\mathbb{E}[X^4] = 2$. Show that
$$\mathbb{P}(X \geq 1) \leq \frac{14}{27}.$$

2. Assume that $X_1, X_2, \dots, X_n, \dots$ is a sequence of iid random variables such that for some $\alpha < 1/2$,
$$\frac{X_1 + X_2 + \cdots + X_n}{n^\alpha} \xrightarrow[n\to\infty]{a.s.} m$$
for some real number m (and the convergence is in the almost sure sense). Show that almost surely $X_i = 0$.

3. Assume that (X, Y) is a joint normal vector with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Show that
$$\mathbb{E}[X^2 Y^2] \geq \mathbb{E}[X^2]\mathbb{E}[Y^2]$$
with equality if and only if X and Y are independent.

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(\mathcal{F})_{n\geq 0}$ is a filtration on it. Show that if $(M_n)_{n\geq 0}$ is a martingale such that $(M_n^4)_{n\geq 0}$ is also a martingale, then almost surely $M_n = M_0$ for any $n \geq 0$.

5. For a sequence $X_1, X_2, \dots, X_n, \dots$ we know that
$$\sum_{n=1}^{\infty} n\mathbb{E}[|X_n|] < \infty.$$
Show that the sequence $Y_n = X_n + X_{n+1} + \cdots + X_{10n}$, converges almost surely and in L^1 to 0.

6. Assume $\{X_n\}_{n\geq 1}$ is a sequence of iid random variables with mean 0 and variance 1. Show that
$$Y_n = \frac{\sqrt{n}X_1 + \sqrt{n-1}X_2 + \sqrt{n-2}X_3 + \cdots + X_n}{n}$$
converges weakly (in distribution) to a normal $N(0, 1/2)$.

7. Assume that $\{U_n\}_{n\geq 1}$ is a sequence of iid uniform random variables on $[0,1]$. Let $V_n = \max\{U_1, U_2^2, \ldots, U_n^n\}$. Show that $(1-V_n)\ln(n)$ converges weakly (in distribution) to an exponential random variable with parameter 1.

8. Let $\{X_n\}_{n\geq 1}$ be an iid sequence of positive random variables such that $E[X_1] < \infty$. Let
$$N_t = \sup\{n : X_1 + X_2 + \cdots + X_n \leq t\}.$$
Show that
$$\frac{N_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1}{\mathbb{E}[X_1]}$$
where the convergence is in almost sure sense.