1. Let \( g : \mathbb{P}^n \to \mathbb{P}^m \) be a map from real projective space of dimension \( n \) to that of dimension \( m \) and denote by \( p_n,p_m \) the 2-fold covering maps \( S^n \to \mathbb{P}^n \) and \( S^m \to \mathbb{P}^m \) respectively. Assume \( n>1, m>0 \).

(a) Prove that there is a map \( f : S^n \to S^m \) such that \( p_m f = gp_n \). Further show that either \( f(-x) = f(x) \) for all \( x \) (“\( f \) is even”) or \( f(-x) = -f(x) \) for all \( x \) (“\( f \) is odd”).

(b) Prove that the function \( f \) in (a) is even precisely when the induced map on fundamental groups \( g_* : \pi_1(\mathbb{P}^n) \to \pi_1(\mathbb{P}^m) \) is trivial and is an odd function precisely when \( g_* \) is an isomorphism.

(c) Use (b) and the fact, which you may assume, that an odd map has odd degree, to prove that when \( n>m \), then \( g_* \) is always the trivial homomorphism.

2. Let \( T \) be the torus \( S^1 \times S^1 \) and \( f : S^1 \to T : \theta \mapsto (\theta,(1,0)) \) for the point \((1,0) \in S^1\). Let \( X \) be the space obtained by attaching a 2-cell \( D^2 \) to \( T \) with the map \( f \).

1. Let \( S^2 \) be the 2-sphere. Show there exists maps \( \phi : S^2 \to X \) and \( \psi : X \to S^2 \) both of which are not homotopic to a constant map. (Hint: consider their composition and degree theory.)

2. Show that any map \( S^2 \) to \( T \) is homotopic to a constant map.

3. Let \( \Sigma \) be a smooth submanifold of \( \mathbb{R}^n \) of co-dimension bigger than 2. Show that \( \mathbb{R}^n - \Sigma \) is connected and simply connected (recall this means that any continuous map of \( S^1 \) into the space is homotopic to a constant loop).

4. Let \( (X,x_0) \) and \( (Y,y_0) \) be path-connected, locally path-connected, and semi-locally simply connected, pointed topological spaces. Let \( f : (X,x_0) \to (Y,y_0) \) be a continuous map. Show that \( f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0) \) is surjective if for any path connected pointed covering map \( p : (E,e_0) \to (Y,y_0) \), the pull-back \((E \times_Y X,e_0 \times x_0) \to (X,x_0) \) is path connected, where \( E \times_Y X = \{(e,x) \in E \times X : f(x) = p(e)\} \).
5. Let $M$ be an $n - 1$ dimensional compact submanifold of $\mathbb{R}^n$ not containing the origin. Show that for almost all directions $v$ in the unit sphere, the ray $\{vt : t \in \mathbb{R}_{\geq 0}\}$ intersects $M$ in only finitely many points.

6. Let $\omega$ be a 1-form on a 3-dimensional manifold $M$. Suppose that $\omega$ is not zero at any point so for each $x \in M$ the kernel $\xi_x$ of $\omega(x)$ is a plane in $T_xM$. We say that $\xi$ is integrable if for any two vector fields $v$ and $w$ with values in $\xi$ (that is $v$ and $w$ are sections of $\xi$) we have that the Lie bracket $[v, w]$ is also a section of $\xi$. For this problem assume that $\omega$ is integrable.

1. Show that $\omega \wedge d\omega = 0$.
2. Show there exists a 1-form $\alpha$ such that $d\omega = \omega \wedge \alpha$. (Hint: prove this locally and then use a partition of unity.)
3. Show that $\omega \wedge d\alpha = 0$.
4. If $\beta$ is another 1-form such that $d\omega = \omega \wedge \beta$ then there is a function $f$ such that $\beta = \alpha + f\omega$ and $\alpha \wedge d\alpha = \beta \wedge d\beta$.

7. Consider the form $\alpha = (x^2 + y^2 + z^2)^{-3/2}(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy)$ on $\mathbb{R}^3 - \{(0, 0, 0)\}$ with Euclidean coordinates $(x, y, z)$. Let $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$.

1. Show that $\alpha$ is closed on $\mathbb{R}^3 - \{(0, 0, 0)\}$.
2. Compute $\int_{S^2} \alpha$.
3. Show $\alpha$ is closed but not exact on $S^2$.
4. Let $\Sigma$ be any compact surface embedded in $\mathbb{R}^3 - \{(0, 0, 0)\}$. What are all the possible values of $\int_{\Sigma} \alpha$. Prove your answer. (You may use the fact that such a surface bounds a compact region $K$ in $\mathbb{R}^3$.)

8. Define a homomorphism $\phi : \mathbb{Z}/2 * \mathbb{Z}/2 \to \Sigma_4$ from the free product of two copies of $\mathbb{Z}/2$, the first with non-zero element called $a$ and the second with the non-zero element called $b$, to the permutation group on the set $\{1, 2, 3, 4\}$ by

\[
\phi(a) = (2, 3)
\]

\[
\phi(b) = (1, 2)(3, 4).
\]

Let $H$ be the subgroup of $\mathbb{Z}/2 * \mathbb{Z}/2$ whose image under $\phi$ stabilizes 1, i.e. $H = \{x : \phi(x)1 = 1\}$. Use covering space theory to find the index of $H$ in its normalizer.