Algebra Comprehensive Exam
Spring 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let R be a commutative ring with 1 and M an ideal of R.
   (a) Show that, if M is both maximal and principal, then there is no ideal I of R such that $M^2 \subsetneq I \subsetneq M$.
   (b) Give an example of a commutative ring R with 1, a maximal ideal M (but not necessarily principal) of R and an ideal I with $M^2 \subsetneq I \subsetneq M$.

2. Let K be a finite extension of a field F, and let P be a monic irreducible polynomial in $K[x]$. Prove that there is a nonzero $Q \in K[x]$ such that the product $P \cdot Q$ is in $F[x]$.

3. Let G be a finite group and let p be a prime number. Show that the following conditions are equivalent:
   (a) The group G acts transitively on some set X such that the cardinality of X is at least 2 and relatively prime to p.
   (b) The order of G is not a power of p.

4. Let R be a commutative ring. An element x is said to be *nilpotent* if $x^k = 0$ for some non-negative integer k. Let P be the set of nilpotent elements. Show that P is an ideal, and that R/P has no non-zero nilpotent elements.

5. How many isomorphism classes of abelian groups of order 6^4 are there? Explain your answer.

6. Is there an injective field homomorphism from $\mathbf{F}_4$ to $\mathbf{F}_{16}$? Is there an injective fields homomorphism from $\mathbf{F}_9$ to $\mathbf{F}_{27}$? Explain your answer.

7. Let M be an $n \times n$ matrix.
   (a) Show that M is invertible if and only if its characteristic polynomial has a non-zero constant term.
   (b) Show that if M is invertible, then its inverse may be expressed as a polynomial in M.

8. Let $f = x^5 - 12x + 6 \in \mathbf{Q}[x]$, and let G be the Galois group of its splitting field.
   (a) Show that f is irreducible, and conclude that $|G|$ is divisible by 5.
   (b) Show that G contains a transposition (hint: complex conjugation).
   (c) Prove that $G = S_5$, and conclude that f is not solvable by radicals.