Student Number: 

*Instructions:* Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.
1. Let $R$ be a commutative ring with 1 and $M$ an ideal of $R$.
   (a) Show that, if $M$ is both maximal and principal, then there is no ideal $I$ of $R$ such
   that $M^2 \subset I \subset M$.
   (b) Give an example of a commutative ring $R$ with 1, a maximal ideal $M$ (but not neces-
   sarily principal) of $R$ and an ideal $I$ with $M^2 \subset I \subset M$.

2. Let $K$ be a finite extension of a field $F$, and let $P$ be a monic irreducible polynomial in $K[x]$. Prove that there is a nonzero $Q \in K[x]$ such that the product $P \cdot Q$ is in $F[x]$.

3. Let $G$ be a finite group and let $p$ be a prime number. Show that the following conditions
   are equivalent:
   (a) The group $G$ acts transitively on some set $X$ such that the cardinality of $X$ is at
   least 2 and relatively prime to $p$.
   (b) The order of $G$ is not a power of $p$.

4. Let $R$ be a commutative ring. An element $x$ is said to be nilpotent if $x^k = 0$ for some
   non-negative integer $k$. Let $P$ be the set of nilpotent elements. Show that $P$ is an ideal,
   and that $R/P$ has no non-zero nilpotent elements.

5. How many isomorphism classes of abelian groups of order $6^4$ are there? Explain your
   answer.

6. Is there an injective field homomorphism from $F_4$ to $F_{16}$? Is there an injective fields
   homomorphism from $F_9$ to $F_{27}$? Explain your answer.

7. Let $M$ be an $n \times n$ matrix.
   (a) Show that $M$ is invertible if and only if its characteristic polynomial has a non-zero
   constant term.
   (b) Show that if $M$ is invertible, then its inverse may be expressed as a polynomial in $M$.

8. Let $f = x^5 - 12x + 6 \in \mathbb{Q}[x]$, and let $G$ be the Galois group of its splitting field.
   (a) Show that $f$ is irreducible, and conclude that $|G|$ is divisible by 5.
   (b) Show that $G$ contains a transposition (hint: complex conjugation).
   (c) Prove that $G = S_5$, and conclude that $f$ is not solvable by radicals.