Algebra Comprehensive Exam
September 2, 2016

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Which of the following groups are isomorphic? Justify your answers
   (i) the multiplicative group of units in $\mathbb{Z}[i]$ where $i^2 = -1$
   (ii) the abelian group generated by a, b, c with relations $a^2 = c^5, a^2 = b^4c^4$, and $b^2 = c$.
   (iii) the subgroup of S_4 generated by $(1\,2)(3\,4)$ and $(1\,3)(2\,4)$
   (iv) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/20\mathbb{Z})$

2. Let R be an integral domain containing a field F. Show that if R has finite dimension as a vector space over F, then R is a field.

3. Let R be an integral domain. Suppose r is a nonzero, non-unit, irreducible element of R, and let $\langle r\rangle$ denote the ideal generated by r.
   (a) If R is a UFD, is $R/\langle r\rangle$ also a UFD?
   (b) If R is a PID, is $R/\langle r\rangle$ also a PID?

4. Let K/F be a Galois extension whose Galois group is the symmetric group S_3. Is it true that K is the splitting field of an irreducible cubic polynomial over F?

5. An *algebraic integer* is the solution to a monic polynomial with coefficients in $\mathbb{Z}$.
   (a) Show that α is an algebraic integer if and only if $\{1, \alpha, \alpha^2, \ldots\}$ generates a finite rank $\mathbb{Z}$-module.
   (b) Let α be an algebraic integer and let $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ be a monic polynomial with coefficients in $\mathbb{Z}$ which has α as a root and is irreducible in $\mathbb{Z}[x]$. Let $R = \mathbb{Z}[\alpha]$. Prove that α is a unit in R if and only if $a_0 = \pm 1$. (Hint: consider $1/x^n(x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0)$.)

6. A *graded ring* is a ring R expressed as a direct sum of modules $R \cong \oplus_{i\in\mathbb{Z}} R_i$ such that the multiplication determines maps $R_i \otimes R_j \to R_{i+j}$. Elements of R_i for some i are called *homogeneous*. Let R be a graded ring such that every nonzero homogeneous element is invertible. Prove that either R is a field concentrated in degree 0 or $R \cong k[\beta^{\pm 1}]$, where k is field.

7. If G is a group acting on a set S, we say that G is n-*transitive* if $|S| \geq n$ and whenever $x_1, \ldots, x_n$ are distinct elements of S and $y_1, \ldots, y_n$ are distinct elements of S, there exists g in G such that $g(x_i) = y_i$ for all $i = 1, \ldots, n$. We denote by S^g the number of fixed points of g. Prove that G is 3-transitive if and only if
$$\frac{1}{|G|}\Sigma_{g\in G}(S^g)^3 = 5.$$

8. Let V be a finite-dimensional vector space over a field F of characteristic p and let $T : V \to V$ be a linear transformation such that $T^p = I$ is the identity map.

   (a) Show that T has an eigenvector in V.

   (b) Show that T is upper triangular with respect to a suitable basis of V.