Analysis Comprehensive Exam
Spring 2019

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.

NOTES:

• $\|x\|$ denotes the Euclidean norm of a point $x \in \mathbb{R}^d$.

• All functions in this exam are (extended) real-valued.

• The exterior Lebesgue measure of $E \subseteq \mathbb{R}^d$ is denoted by $|E|_e$, and if $E$ is measurable then its Lebesgue measure is $|E|$.

• The characteristic function of a set $A$ is denoted by $\chi_A$. 

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1. For \( n \geq 1 \), let \( f_n : [0, 1] \to \mathbb{R} \) be integrable. Assume that 
\[
\lim_{k \to \infty} f_k = f \text{ a.e. in } [0, 1],
\]
where \( f \) is integrable over \([0, 1]\). Assume also that \( \forall \varepsilon > 0 \), there exists \( \delta > 0 \) such that 
\[
E \subset [0, 1] \text{ and } |E| < \delta \Rightarrow \left| \int_E f_k \right| < \varepsilon \text{ for all } k \geq 1.
\]
Prove that 
\[
\lim_{k \to \infty} \int_0^1 |f_k - f| = 0. \quad (1)
\]

2. (a) Assume that \( \mu \) is a bounded linear functional on \( L^2(\mathbb{R}) \). Prove directly that 
\( F(x) = \mu(\chi_{[0,x]}) \) is absolutely continuous on \([0, 1]\). (Directly means that you should not appeal to the Riesz Representation Theorem in this part.)

(b) Use the Riesz Representation Theorem to find a formula for \( F'(x) \) that holds for a.e. \( x \).

3. Let \( \mu \) and \( \nu \) be Borel measures on \([0, \infty)\) with finite total mass, so that \( \mu([0, \infty)) < \infty \) and \( \nu([0, \infty)) < \infty \). Let \( r \in (0, 1) \), \( s > 0 \) and \( \omega \) be the measure defined by 
\[
\omega = r\mu + s\nu.
\]
Show that \( \mu \) is absolutely continuous with respect to \( \omega \). Let \( g \) denote the Radon-Nikodym derivative of \( \mu \) with respect to \( \omega \), so that 
\[
\int f \, d\mu = \int fg \, d\omega
\]
for every integrable function \( f \). Show that 
\[
0 \leq g \leq \frac{1}{r} \text{ a.e. } \mu
\]

4. Assume \( E \subseteq \mathbb{R}^d \) is measurable, \( f : E \to [0, \infty) \) is measurable and finite a.e., and \( g : [0, \infty) \to [0, \infty) \) is absolutely continuous on every finite interval \([0, b]\) and is monotone increasing on \([0, \infty)\). Prove that 
\[
\int_E g \circ f \geq \int_0^\infty g'(t) \omega(t) \, dt,
\]
where \( \omega(t) = |\{f > t\}|. \)

Hint: First show that \( \int_0^{f(x)} g'(t) \, dt \leq g(f(x)) \).
5. Let \( \phi : \mathbb{Z} \to (0, \infty) \), that is, \( \phi \) is a positive function defined on the integers. Assume also that
\[
\sum_{k=1}^{\infty} k^2 \phi(k)^2 < \infty.
\]
Let \( \mathcal{A} \subseteq \mathbb{R}^2 \) be the set of all \((x, y) \in \mathbb{R}^2\) such that for infinitely many \( k \geq 1 \), there exist a pair of rational numbers \( \left( \frac{j}{k}, \frac{\ell}{k} \right) \) with
\[
\left| (x, y) - \left( \frac{j}{k}, \frac{\ell}{k} \right) \right| < \phi(k). \tag{1}
\]
Show that \(|\mathcal{A}| = 0\).

6. Let \( X \) and \( Y \) be Banach spaces. Suppose that \( A : S \to Y \) is a bounded linear operator whose domain \( S \) is a dense subspace of \( X \). Prove that there exists a unique bounded linear operator \( B : X \to Y \) such that \( B(x) = A(x) \) for all \( x \in S \). Show further that the operator norm of \( B \) equals the operator norm of \( A \).

7. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be integrable in \( \mathbb{R}^n \). Let \( K : \mathbb{R}^n \to [0, \infty) \) be nonnegative, measurable, and bounded in \( \mathbb{R}^n \), with
\[
\int_{\mathbb{R}^n} K = 1
\]
and \( K(t) = 0 \) for \( |t| \geq 1 \). For \( h > 0 \), and \( x \in \mathbb{R}^n \), define
\[
\Phi_h[f](x) = h^{-n} \int_{\mathbb{R}^n} f(x + t) K\left( \frac{t}{h} \right) dt.
\]
For \( h > 0 \), let
\[
\Omega(f; h) = \sup_{|t| \leq h} \int_{\mathbb{R}^n} |f(x + t) - f(x)| dx.
\]
Prove that
\[
\int_{\mathbb{R}^n} |\Phi_h[f](x) - f(x)| \, dx \leq \Omega(f; h),
\]
and hence that
\[
\lim_{h \to 0^+} \int_{\mathbb{R}^n} |\Phi_h[f](x) - f(x)| \, dx = 0.
\]

8. Given \( f \in L^2(0, \infty) \), prove that
\[
F(x) = \int_0^\infty \frac{f(t)}{1 + xt} dt
\]
is continuous and differentiable on \((0, \infty)\).