Analysis Comprehensive Exam
Spring 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTES:

- $\|x\|$ denotes the Euclidean norm of a point $x \in \mathbf{R}^d$.
- All functions in this exam are (extended) real-valued.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.

1. For $n \geq 1$, let $f_n : [0,1] \to \mathbf{R}$ be integrable. Assume that

$$\lim_{k\to\infty} f_k = f \text{ a.e. in } [0,1],$$

where f is integrable over $[0,1]$. Assume also that $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$E \subset [0,1] \text{ and } |E| < \delta \;\Rightarrow\; \left|\int_E f_k\right| < \varepsilon \text{ for all } k \geq 1.$$

Prove that

$$\lim_{k\to\infty} \int_0^1 |f_k - f| = 0. \tag{1}$$

2. (a) Assume that μ is a bounded linear functional on $L^2(\mathbf{R})$. Prove directly that $F(x) = \mu(\chi_{[0,x]})$ is absolutely continuous on $[0,1]$. (Directly means that you should not appeal to the Riesz Representation Theorem in this part.)

(b) Use the Riesz Representation Theorem to find a formula for $F'(x)$ that holds for a.e. x.

3. Let μ and ν be Borel measures on $[0,\infty)$ with finite total mass, so that $\mu([0,\infty)) < \infty$ and $\nu([0,\infty)) < \infty$. Let $r \in (0,1)$, $s > 0$ and ω be the measure defined by

$$\omega = r\mu + s\nu.$$

Show that μ is absolutely continuous with respect to ω. Let g denote the Radon-Nikodym derivative of μ with respect to ω, so that

$$\int f\, d\mu = \int fg\, d\omega$$

for every integrable function f. Show that

$$0 \leq g \leq \frac{1}{r} \text{ a.e. } (\mu)$$

4. Assume $E \subseteq \mathbf{R}^d$ is measurable, $f\colon E \to [0,\infty)$ is measurable and finite a.e., and $g\colon [0,\infty) \to [0,\infty)$ is absolutely continuous on every finite interval $[0,b]$ and is monotone increasing on $[0,\infty)$. Prove that

$$\int_E g\circ f \geq \int_0^\infty g'(t)\,\omega(t)\,dt,$$

where $\omega(t) = \big|\{f > t\}\big|$.

Hint: First show that $\int_0^{f(x)} g'(t)\,dt \leq g(f(x))$.

5. Let $\phi : \mathbf{Z} \to (0, \infty)$, that is, ϕ is a positive function defined on the integers. Assume also that
$$\sum_{k=1}^{\infty} k^2 \phi(k)^2 < \infty.$$
Let $\mathcal{A} \subset \mathbf{R}^2$ be the set of all $(x, y) \in \mathbf{R}^2$ such that for infinitely many $k \geq 1$, there exist a pair of rational numbers $\left(\frac{j}{k}, \frac{\ell}{k}\right)$ with
$$\left| (x, y) - \left(\frac{j}{k}, \frac{\ell}{k} \right) \right| < \phi(k). \tag{1}$$
Show that $|\mathcal{A}| = 0$.

6. Let X and Y be Banach spaces. Suppose that $A \colon S \to Y$ is a bounded linear operator whose domain S is a dense subspace of X. Prove that there exists a unique bounded linear operator $B \colon X \to Y$ such that $B(x) = A(x)$ for all $x \in S$. Show further that the operator norm of B equals the operator norm of A.

7. Let $f : \mathbf{R}^n \to \mathbf{R}$ be integrable in $\mathbf{R}^n$. Let $K : \mathbf{R}^n \to [0, \infty)$ be nonnegative, measurable, and bounded in $\mathbf{R}^n$, with
$$\int_{\mathbf{R}^n} K = 1$$
and $K(\mathbf{t}) = 0$ for $|\mathbf{t}| \geq 1$. For $h > 0$, and $\mathbf{x} \in \mathbf{R}^n$, define
$$\Phi_h[f](\mathbf{x}) = h^{-n} \int_{\mathbf{R}^n} f(\mathbf{x} + \mathbf{t}) K\left(\frac{\mathbf{t}}{h}\right) d\mathbf{t}.$$
For $h > 0$, let
$$\Omega(f; h) = \sup_{|\mathbf{t}| \leq h} \int_{\mathbf{R}^n} |f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})| \, d\mathbf{x}.$$
Prove that
$$\int_{\mathbf{R}^n} |\Phi_h[f](\mathbf{x}) - f(\mathbf{x})| \, d\mathbf{x} \leq \Omega(f; h),$$
and hence that
$$\lim_{h \to 0+} \int_{\mathbf{R}^n} |\Phi_h[f](\mathbf{x}) - f(\mathbf{x})| \, d\mathbf{x} = \mathbf{0}.$$

8. Given $f \in L^2(0, \infty)$, prove that
$$F(x) = \int_0^{\infty} \frac{f(t)}{1 + xt} \, dt$$
is continuous and differentiable on $(0, \infty)$.