Analysis Comprehensive Exam
August 26, 2016

Student Number: [___]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.

Notations used throughout the exam:

For $E \subseteq \mathbb{R}^d$, the exterior Lebesgue measure of $E$ is written $|E|_e$.

If $E$ is measurable then its Lebesgue measure is denoted $|E|$.

We denote the dual space of a Banach space $V$ by $V^*$, i.e. $V^*$ is the collection of all bounded linear functionals acting on $V$. 
1. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $0 < |E| < \infty$.

   (i) For each $x \in \mathbb{R}$ and $r > 0$ define $I_r(x) = [x - r/2, x + r/2]$ and $h_r(x) = |E \cap I_r(x)|$. Prove that for a fixed $r > 0$, the function $h_r(x)$ is continuous at every $x \in \mathbb{R}$.

   (ii) Prove that there exists $r_0 > 0$ such that for each $0 < r < r_0$ there exists a closed interval $I \subset \mathbb{R}$ which satisfies $|I| = r$ and $|E \cap I| = r/2$.

2. Show that for $A \subset \mathbb{R}^d$, $A$ is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists a Lebesgue measurable set $E \subset \mathbb{R}^d$ such that

   $$|A \triangle E|_\epsilon < \epsilon.$$  

3. Let $E_1, \ldots, E_n$ be Lebesgue measurable subsets of $[0, 1]$ and define

   $$S_q = \{x \in [0, 1] : x \text{ belongs to at least } q \text{ of the sets } E_i\}.$$  

   Show that for each $1 \leq q \leq n$, $S_q$ is Lebesgue measurable and there exists $k$ such that

   $$\frac{q |S_q|}{n} \leq |E_k|.$$  

4. Prove that if $f(x), xf(x) \in L^1(\mathbb{R})$ then the function

   $$F(w) = \int_{\mathbb{R}} f(x) \sin(wx) \, dx$$

   is defined, continuous, and differentiable at every point $w \in \mathbb{R}$. (You may wish to use the identity $\sin(\alpha) - \sin(\beta) = 2 \sin(\frac{\alpha - \beta}{2}) \cos(\frac{\alpha + \beta}{2})$).

5. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Given $y > 0$ denote $A_y := \{x \in \mathbb{R} : |f(x)| > y\}$. Prove that

   $$\int_{\mathbb{R}} |f(x)|^p \, dx = p \int_0^\infty y^{p-1} |A_y| \, dy.$$  

6. Let $\mu$ and $\nu$ be two $\sigma$-finite positive measures on a measurable space $(X, \mathcal{M})$. Show that there exists a measurable function $f : X \to \mathbb{R}$ such that for each $E \in \mathcal{M}$,

   $$\int_E (1 - f) \, d\mu = \int_E f \, d\nu.$$  

   Does the above statement hold for every finite signed measures $\mu$ and $\nu$?
7. Let $\mathcal{H}$ be a Hilbert space and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}$. Prove that the following two statements are equivalent

(i) There exists $C > 0$ such that for every $f \in \mathcal{H}$,
$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq C \|f\|^2.$$

(ii) There exists $C > 0$ such that for every sequence $\{a_n\}_{n \in \mathbb{N}}$ with finitely many nonzero terms
$$\left\| \sum_{n=1}^{\infty} a_n f_n \right\|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2.$$

8. Let $V$ be a Banach space and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $V$. For $m \in \mathbb{N}$ let
$$W_m = \text{span}\{f_n\}_{n \neq m}.$$ 
Prove that the following two statements are equivalent.

(i) There exists $d > 0$ such that for every $m \in \mathbb{N}$,
$$d \leq \text{dist}(f_m, W_m).$$

(ii) There exist $M > 0$ and a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $V^*$ such that for every $n \in \mathbb{N}$ we have $\|g_n\|_{V^*} < M$ and for each $m \in \mathbb{N}$,
$$g_n(f_m) = \delta_{nm}.$$