# Analysis Comprehensive Exam
# Fall 2018

**Student Number:** ☐

*Instructions:* Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTES:

- All functions in this exam are (extended) real-valued.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if $E$ is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set $A$ is denoted by $\chi_A$.

1. For each irrational number $x$, it can be shown that there exist infinitely many fractions $p/q$ with $p$ and $q$ relatively prime such that $|x - p/q| \le 1/q^2$. Let $E$ be the set of all $x \in \mathbf{R}$ for which there exist infinitely many fractions $p/q$ with $p$ and $q$ relatively prime such that
$$\left|x - \frac{p}{q}\right| \le \frac{1}{q^3}.$$
Prove that $|E| = 0$.

2. Suppose that $f \in L^1(\mathbf{R})$ is such that $\int_{-\infty}^{\infty} f\phi = 0$ for every integrable simple function $\phi$ that satisfies $\int_{-\infty}^{\infty} \phi = 0$. Prove that $f = 0$ a.e.

3. Choose $f \in L^1(\mathbf{R})$, and for each $\lambda > 0$ define $f_\lambda(x) = \lambda f(\lambda x)$. Prove that
$$\lim_{\lambda \to 1} \|f - f_\lambda\|_1 = 0.$$

4. Let $f$ be an integrable function defined on a measurable set $E \subseteq \mathbf{R}^d$. Show that if $\{A_n\}_{n \in \mathbf{N}}$ is a sequence of measurable subsets of $E$ such that $|A_n| \to 0$, then $\int_{A_n} f \to 0$.

5. Let $f \colon \mathbf{R} \to \mathbf{R}$ be absolutely continuous on every finite interval $[a, b]$. Assume that $f' \in L^2(\mathbf{R})$ and $f \in L^2(\mathbf{R})$, and prove that $\lim_{x \to \pm\infty} f(x) = 0$.

6. (a) Given $t > 0$, use Fubini's theorem to compute $\displaystyle\int_0^t \int_0^\infty e^{-xy} \sin(x)\, dy\, dx$ in two different ways, and then use this to compute the exact value of $\displaystyle\lim_{t \to \infty} \int_0^t \frac{\sin(x)}{x} dx$.

   Hint: $\displaystyle\int_0^\infty \frac{1}{1+y^2}\, dy = \frac{\pi}{2}$.

   (b) Show that the function $f(x) = \sin(x)/x$ is not Lebesgue integrable on $[0, \infty)$.

   (c) Explain why part (b) does not affect the applicability of Fubini's theorem in (a).

7. Let $E \subseteq \mathbf{R}^d$ be measurable with $0 < |E| < \infty$. Assume that:

   (a) $f_n \in L^1(E)$ for every $n$,

   (b) there exists a function $f$ such that $f_n \to f$ pointwise a.e., and

   (c) there exists some $1 < p < \infty$ such that $\sup_n \|f_n\|_p < \infty$.

   Prove that $f \in L^1(E)$ and $f_n \to f$ in $L^1$-norm.

8. Let $H$ be a (complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $a$ be a fixed complex number, and assume that $T \colon H \to H$ is a linear operator that satisfies
$$\langle x, Ty \rangle = a\, \langle Tx, y \rangle, \qquad \text{all } x,\, y \in H.$$

Show that $T$ is bounded.

Hint: The case $|a| \neq 1$ is easy.

Note: The case $a = 1$ is related to the Hellinger–Toeplitz theorem, but you must provide a direct proof, you cannot appeal to that theorem.