Discrete Mathematics Comprehensive Exam
Fall 2018

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a graph and $V_o = \{v \in V(G) \ : \ d_G(v) = 1 \pmod 2\}$. Show that G has $|V_o|/2$ pairwise edge-disjoint paths linking vertices in V_0.

2. Let G be a graph and k be a positive integer. Show that $\chi(G) \leq k$ if, and only if, G has an acyclic orientation with no directed path of length k.

3. Suppose that G is a plane graph and assume that G contains a Hamilton cycle. Prove that the faces of G are 4-colorable.

4. Prove that every triangle-free graph G with n vertices and m edges has a bipartite subgraph with at least $\frac{4m^2}{n^2}$ edges. (Hint: Consider $G - x - N(x)$ for $x \in V(G)$.)

5. Fix $k \geq 2$. Using a *probabilistic approach*, prove that if G is a graph with $m \geq 1$ edges then G has a k-partite subgraph with at least $\frac{k-1}{k} \cdot m$ edges.

6. Let $G_{n,p}$ be the binomial random graph with vertex set $[n]$, where each of the $\binom{n}{2}$ edges of the complete graph K_n is included, independently, with probability p. Given a graph H, let $\alpha(H)$ denote the size of the largest independent set of H. Given $p = p(n) \in (0, 1]$, prove that $\mathbb{P}(\alpha(G_{n,p}) < 6 \ln n/p) \to 1$ as $n \to \infty$.

7. Let $G_{n,p}$ be the binomial random graph with vertex set $[n]$, where each of the $\binom{n}{2}$ edges of the complete graph K_n is included, independently, with probability p. Let $\mathcal{E}_k$ denote the event that $G_{n,p}$ contains some collection of k edge-disjoint triangles. Give a complete proof of the estimate $\mathbb{P}(\mathcal{E}_k) \leq \left(\binom{n}{3}p^3\right)^k/k!$
(Remark: only submit a solution to the aforementioned estimate. If you have an idea how to improve this estimate for $k = x\binom{n}{3}p^3$ with $x \in (1, e)$, please email `warnke@math.gatech.edu` after the exam.)