Discrete Mathematics Comprehensive Exam
Fall 2018

Student Number: 

Instructions: Complete 5 of the 7 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $G$ be a graph and $V_0 = \{v \in V(G) : d_G(v) = 1 \text{ (mod 2)}\}$. Show that $G$ has $|V_0|/2$ pairwise edge-disjoint paths linking vertices in $V_0$.

2. Let $G$ be a graph and $k$ be a positive integer. Show that $\chi(G) \leq k$ if, and only if, $G$ has an acyclic orientation with no directed path of length $k$.

3. Suppose that $G$ is a plane graph and assume that $G$ contains a Hamilton cycle. Prove that the faces of $G$ are 4-colorable.

4. Prove that every triangle-free graph $G$ with $n$ vertices and $m$ edges has a bipartite subgraph with at least $\frac{4nm^2}{n^2}$ edges. (Hint: Consider $G - x - N(x)$ for $x \in V(G)$.)

5. Fix $k \geq 2$. Using a probabilistic approach, prove that if $G$ is a graph with $m \geq 1$ edges then $G$ has a $k$-partite subgraph with at least $\frac{k-1}{k} \cdot m$ edges.

6. Let $G_{n,p}$ be the binomial random graph with vertex set $[n]$, where each of the $\binom{n}{2}$ edges of the complete graph $K_n$ is included, independently, with probability $p$. Given a graph $H$, let $\alpha(H)$ denote the size of the largest independent set of $H$. Given $p = p(n) \in (0, 1]$, prove that $\mathbb{P}(\alpha(G_{n,p}) < 6 \ln \frac{n}{p}) \to 1$ as $n \to \infty$.

7. Let $G_{n,p}$ be the binomial random graph with vertex set $[n]$, where each of the $\binom{n}{2}$ edges of the complete graph $K_n$ is included, independently, with probability $p$. Let $\mathcal{E}_k$ denote the event that $G_{n,p}$ contains some collection of $k$ edge-disjoint triangles. Give a complete proof of the estimate $\mathbb{P}(\mathcal{E}_k) \leq \left( \binom{n}{3} p^3 \right)^k / k!$

(Remark: only submit a solution to the aforementioned estimate. If you have an idea how to improve this estimate for $k = x \binom{n}{3} p^3$ with $x \in (1, e)$, please email warnke@math.gatech.edu after the exam.)