Discrete Mathematics Comprehensive Exam
Spring 2019

Student Number: [ ]

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let k be a positive integer and G be a simple graph, and assume that G does not contain k pairwise disjoint cycles. Show that $\chi(G) \le 3k - 1$.

2. Let G and H be two simple graphs and assume $\Delta(H) \le 3$. Show that if G contains H as a minor then G contains H as a topological minor.

3. Let G be a graph and assume that G contains a Hamilton cycle. Show that G can be written as the union of two even subgraphs (not necessarily edge-disjoint).

4. Let G be a maximal plane graph and $G \not\cong K_4$. Show that G is 3-face-colorable.

5. The Ramsey number $R(k)$ is the smallest integer n such that in any two coloring of the edges of a complete graph on n vertices K_n by red and blue contains a monochromatic K_k (a complete subgraph on k vertices whose edges are all colored with the same color).

   (A) Show that if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k) > n$.
   (B) Using (A), deduce that $R(k) = \Omega(k2^{k/2})$ for large k.
   [Remark: this argument in fact gives $R(k) > (1 - o(1)) \cdot k2^{k/2} \cdot 1/(e\sqrt{2})$ as $k \to \infty$.]

6. The Ramsey number $R(k)$ is the smallest integer n such that in any two coloring of the edges of a complete graph on n vertices K_n by red and blue contains a monochromatic K_k (a complete subgraph on k vertices whose edges are all colored with the same color).

   (A) Show that $R(k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$.
   (B) Using (A), deduce that $R(k) = \Omega(k2^{k/2})$ for large k.
   [Remark: this argument in fact gives $R(k) > (1 - o(1)) \cdot k2^{k/2} \cdot 1/e$ as $k \to \infty$.]

7. The Ramsey number $R(k)$ is the smallest integer n such that in any two coloring of the edges of a complete graph on n vertices K_n by red and blue contains a monochromatic K_k (a complete subgraph on k vertices whose edges are all colored with the same color).

   (A) Show that if $e\binom{k}{2}\binom{n-2}{k-2}2^{1-\binom{k}{2}} < 1$, then $R(k) > n$. (Hint: use LLL.)
   (B) Using (A), deduce that $R(k) = \Omega(k2^{k/2})$ for large k.
   [Remark: this argument in fact gives $R(k) > (1 - o(1)) \cdot k2^{k/2} \cdot \sqrt{2}/e$ as $k \to \infty$.]