Numerical Analysis Comprehensive Exam
Spring 2019

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Derive the composite Simpson's rule to compute the integral

$$I(f) = \int_a^b f(x)dx$$

where $f(x) \in C^\infty$. Derive its error estimate (you must show your steps to receive credit). What is the error of $I(f)$ when $f(x) = x^3 + 65x^2 - 3$?

2. Find a formula for the following polynomial interpolation problem. Let $x_i = x_0 + ih$, $i = 0, 1, 2, 3$, $h > 0$. Find a polynomial $p(x)$ of degree ≤ 5 for which

$$\begin{aligned} p(x_i) &= f(x_i), \quad i = 0, 1, 2, 3; \\ p'(x_0) &= f'(x_0), \quad p'(x_3) = f'(x_3), \end{aligned} \tag{1}$$

where $f(x)$ has continuous derivative of any order in $(-\infty, \infty)$. Derive an error formula for $f(x) - p(x)$. What's the order of approximation for $x \in [x_0, x_3]$?

3. Consider solving the ODE $y'(t) = f(t, y)$, $y(0) = y_0$ numerically by the following multi-step scheme

$$y_{n+3} + (2b - 3)(y_{n+2} - y_{n+1}) - y_n = hb(f_{n+2} + f_{n+1}),$$

where $h > 0$ is the step size, $f_n = f(t_n, y_n)$ and b is a nonzero real number. Determine all values of b so that the scheme is zero-stable. What is the order of accuracy of the scheme when it is zero-stable?

4. Consider the heat equation $\partial_t u = \partial_x^2 u$, $x \in (0, 1)$, with boundary conditions $u(0, t) = g_1(t)$, $u(1, t) = g_2(t)$ and initial value $u(x, 0) = f(x)$, where g_1, g_2 and f are continuous along the space-time boundary. On a uniform grid, the implicit scheme for solving the equation can be written as

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}.$$

Show that the scheme is unconditionally stable in l^∞.

5. Consider the system

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0,$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Find a stable explicit scheme to solve it. Derive the truncation error and stability condition for the scheme.

6. Consider a matrix

$$A = \begin{bmatrix} 1 & 0 & 1+\delta \\ 0, & 1 & 0 \\ 1-\delta & 0 & 1 \end{bmatrix},$$

where $\delta \in [0, 1]$.

(a) What is the condition number of matrix A measured in $\infty-$norm.

(b) Take $\delta = 0.0001$. If one solves $A\vec{x} = \vec{b}$ for a given $\vec{b}$ by Gaussian Elimination with partial pivoting, and makes a perturbation to $\vec{b}$ by $\delta\vec{b}$, with $\frac{\|\delta\vec{b}\|}{\|\vec{b}\|} \leq 10^{-10}$, what is the relative error produced in the solution? The machine precision is 10^{-16}. You must justify your answer.

7. Consider to solve a linear system $A\vec{x} = \vec{b}$ with A being a $m \times m$ symmetric positive definite matrix. Given two subspaces K and L of $\mathbb{R}^m$, one defines the projection method as follows: Given an initial guess $\vec{x}_{n-1}$, find $\vec{x}_n \in \{\vec{x}_{n-1} + K\}$ such as the residual $\vec{r}_n$ is orthogonal to L.

(a) Take $K = L$, then $\vec{x}_n$ is the solution of the projection method if and only if

$$\vec{x}_n = argmin_{\vec{x} \in \{\vec{x}_{n-1}+K\}} (\vec{x}^* - \vec{x})^T A (\vec{x}^* - \vec{x}),$$

where $\vec{x}^*$ is the exact solution of $A\vec{x} = \vec{b}$.

(b) If one selects the subspaces $K = L = span\{\vec{r}\}$, where $\vec{r}$ is the residual of the previous iteration, give an algorithm to implement the projection method (you may write it in pseudo code). What is the computational cost in each iteration of your algorithm?