Numerical Analysis Comprehensive Exam
Fall 2018

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $f(x)$ be a sufficiently smooth function defined on $[a, b]$. show that the Simpson's rule is 5th order accurate, i.e.

$$\int_a^b f(x)dx = \frac{h}{6}\{f(a) + 4f(\frac{a+b}{2}) + f(b)\} + O(h^5),$$

where $h = b - a$.

2. Consider the iteration $x_{n+1} = 2 - (2 + c^2)x_n + cx_n^3$ that converges to $\alpha = 1$ for the value of c (provided x_0 is chosen sufficiently close to α). Find the values of c for which this is true. For what value of c is the convergence quadratic?

3. Consider the following system of differential equations,

$$\begin{cases} \frac{dx(t)}{dt} = -Ax, \\ x(0) = x_0 \end{cases}$$

where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$, with

$$A = \begin{bmatrix} 1 & -0.01 \\ 0 & 1000 \end{bmatrix},$$

   (a) What is the forward Euler scheme for this system of equations? When is the scheme stable?

   (b) Design a scheme that is stable for arbitrary step sizes.

   You must justify your answers.

4. Consider a linear finite difference scheme $U^{n+1} = QU^n$ for a well-posed 1D linear initial value problem $\partial_t u = Lu$, where $U^n = \{\cdots, U_j^n, U_{j+1}^n, \cdots\}$, Q is a linear difference operator and L is a linear differential operator. Suppose the scheme's truncation error is bounded by $C(\Delta x^p + \Delta t^q)$ w.r.t. a norm $||\cdot||$ for some constant $C > 0$ independent of $\Delta x, \Delta t$ and time (as long as it's no larger than the final time T), and is stable w.r.t. the norm as long as $n\Delta t \leq T$. Here p and q are positive integers. Does the numerical solution converge to the smooth solution of the initial value problem w.r.t. the norm at any time in $(0, T)$ as $\Delta x, \Delta t \to 0$? If so, what's the order of convergence. Justify your answer.

5. Consider the heat equation $\partial_t u = \partial_x^2 u + \partial_y^2 u$, $(x, y) \in R^2$. On a uniform rectangular grid, the ADI scheme for solving the equation can be written as

$$\frac{U_{jk}^{n+\frac{1}{2}} - U_{jk}^n}{\Delta t/2} = D_x^2 U_{jk}^{n+\frac{1}{2}} + D_y^2 U_{jk}^n$$

and

$$\frac{U_{jk}^{n+1} - U_{jk}^{n+\frac{1}{2}}}{\Delta t/2} = D_x^2 U_{jk}^{n+\frac{1}{2}} + D_y^2 U_{jk}^{n+1},$$

where $D_x^2 U_{jk}^n = (U_{j+1,k}^n - 2U_{j,k}^n + U_{j-1,k}^n)/\Delta x^2$, similarly for $D_y^2 U_{jk}^n$ and other terms. What is the truncation error of the scheme ? Study its stability for the initial value problem.

6. Let $\vec{w} \in \mathbb{R}^n$ satisfying $\|\vec{w}\|_2 = 1$. Define

$$H = I - 2\vec{w}\vec{w}^T.$$

   (a) Prove that $H^2 = I$, and for any $\vec{x} \in \mathbb{R}^n$, $H\vec{x}$ and $\vec{x}$ are mirror reflections of each other with respect to the hyper-plane that is orthogonal to $\vec{w}$.

   (b) For any given $\vec{x} \neq 0$, construct $\vec{w}$ such that

$$H\vec{x} = -\|\vec{x}\|_2 \vec{e}_1,$$

   where $\vec{e}_1$ is the unit vector $[1, 0, \cdots, 0]^T$.

7. Consider to solve a linear system of equation $A\vec{x} = \vec{b}$ by the following iterative method, where A is a $m \times m$ real symmetric positive definite matrix.

$$\begin{aligned}
&\vec{r} = \vec{b} - A\vec{x}_0;\\
&while \quad \vec{r} \neq 0\\
&\qquad \alpha = \tfrac{\vec{r}^T\vec{r}}{\vec{r}^T A\vec{r}}.\\
&\qquad \vec{x}_n = \vec{x}_{n-1} + \alpha\vec{r};\\
&\qquad \vec{r} = \vec{r} - \alpha A\vec{r};\\
&end;
\end{aligned}$$

   (a) Prove that $\{\vec{x}_n\}$ converges to $\vec{x}^*$, the solution of the linear system, for any initial $\vec{x}_0$.

   (b) If A is the following given matrix,

$$A = \begin{bmatrix} 0.01 & 0.0001 \\ 0.0001 & 1 \end{bmatrix},$$

   estimate the number of iterations needed to obtain

$$\|\vec{x}_n - \vec{x}^*\|_A \leq 10^{-2}\|\vec{x}_0 - \vec{x}^*\|_A,$$

   when using the iterative method in (a).