Probability Comprehensive Exam
Spring 2019

Student Number: ☐

Instructions: Complete 5 of the 9 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X be a non-negative random variable, such that $0 < \mathbb{E}X < +\infty$, and let $0 < x < 1$. Show that
$$\mathbb{P}(X \geq x\mathbb{E}X) \geq (1-x)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)}.$$

2. If $(X_n)_{n\geq 1}$ is a sequence of random variables, then there exists a sequence $(c_n)_{n\geq 1}$ with $c_n \to \infty$, such that
$$\mathbb{P}(\lim_{n\to\infty} \frac{X_n}{c_n} = 0) = 1.$$

3. Assume that $\{X_n\}_{n\geq 1}$ are random variables such that

   1. $E[X_n] = 0$ and $\mathbb{E}[X_n^2] \leq 1$ for any $n \geq 1$
   2. $\mathbb{E}[X_i X_j] \leq 0$ for any $i \neq j$.

   Show that for any sequence $\{a_n\}_{n\geq 1} \subset [1/2, 2]$,
$$\frac{a_1 X_1 + a_2 X_2 + \cdots + a_n X_n}{a_1 + a_2 + \cdots + a_n} \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

4. Let $(X_n)_{n\geq 1}$ be a sequence of non -negative uniformly integrable random variables such that, as $n \to +\infty$, $X_n \Rightarrow X$. Show that X is integrable and that $\lim_{n\to+\infty} \mathbb{E}X_n = \mathbb{E}X$.

5. If $X_1, X_2, \ldots, X_n$ are iid exponential random variables with parameter 1, compute the almost sure limit of
$$\frac{1}{n} \sum_{i=1}^{n} e^{-X_k - 2X_{k+1} - 3X_{k+2}}$$
as n tends to infinity.

6. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space such that there exist $X_1, X_2 : \Omega \to \mathbb{R}$ two independent Bernoulli random variables such that $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = 1/2$. Show that Ω must have at least 4 elements.

   Give an example with Ω having 4 elements together with a sigma algebra such that on it we can define two independent Bernoulli as above.

   Can you generalize this?

7. If X, Y are two random variables such that $X \geq Y$ and X, Y have the same distribution, then $X = Y$ almost surely.

8. Assume that $X_1, X_2, \ldots, X_n$ are iid with density $f(x) = \frac{2}{x^3}$ for $x \geq 1$ and 0 otherwise. Define

$$M_n = \frac{1}{n} \max\{X_1, \sqrt{2}X_2, \ldots, \sqrt{n}X_n\}.$$

Show that X_n converges in distribution and find the limit.

9. Let X be a finite mean random variable, let $\mathbf{F}$ be a σ-field and let G be a σ-field independent of $\sigma(\sigma(X), \mathbf{F})$. (As usual, $\sigma(X)$ is the σ-field generated by X and $\sigma(\sigma(X), \mathbf{F})$ is the σ-field generated by $\sigma(X)$ and $\mathbf{F}$.) Is it true or false that $\mathbb{E}(X|\sigma(\mathbf{F}, \mathbf{G})) = \mathbb{E}(X|\mathbf{F})$?