Probability Comprehensive Exam
Fall 2018

Student Number: ☐

Instructions: Complete 5 of the 9 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Use the SLLN to find the following limit:

$$\lim_{n\to\infty}\int_0^1\cdots\int_0^1 \frac{x_1^2+\cdots+x_n^2}{x_1+\cdots+x_n}dx_1\dots dx_n.$$

2. Suppose $X_1,\dots,X_n$ are i.i.d. random variables such that $\mathbb{P}\{X_j=+1\}=\mathbb{P}\{X_j=-1\}=1/2$. Let $S_k:=X_1+\cdots+X_k$, $k=1,\dots,n$. Prove that

$$\mathbb{P}\{\max_{1\le k\le n} S_k\ge l\}=2\mathbb{P}\{S_n>l\}+\mathbb{P}\{S_n=l\}.$$

3. Let $\{Z_n\}$ be i.i.d. standard normal r.v. and let $\{a_n\}$ be a sequence of nonnegative real numbers. Prove that $\sum_{n=1}^\infty a_nZ_n^2<+\infty$ a.s. if and only if $\sum_{n=1}^\infty a_n<+\infty$.

4. Let φ be the characteristic function of r.v. X. Show that

$$\psi_1(t)=|\varphi(t)|^2 \text{ and } \psi_2(t)=\frac{1}{t}\int_0^t\varphi(s)ds$$

are also characteristic functions.

5. For distribution functions F,G on the real line, define

$$L(F,G):=\inf\Big\{\varepsilon>0:\forall t\in\mathbb{R}\ F(t)\le G(t+\varepsilon)+\varepsilon, G(t)\le F(t+\varepsilon)+\varepsilon\Big\}.$$

It is known that L is a metric. Prove that $L(F_n,F)\to 0$ as $n\to\infty$ if and only if F_n converges weakly to F.

6. Let $X_1,X_2,\dots,X_n,\dots$ be identically distributed (not necessarily independent!) random variables with finite first moment. Is the following,

$$n^{-1}\mathbb{E}\max_{1\le k\le n}|X_k|\longrightarrow 0,$$

as $n\to+\infty$, true or false?

7. Let $X_1,X_2,\dots,X_n,\dots$ be iid random variables with common characteristic function φ and let $S_n=\sum_{k=1}^n X_k$. Show that if φ is differentiable at 0 with $\varphi'(0)=i\mu$, then, as $n\to+\infty$, $S_n/n\to\mu$, in probability.

8. Let X and Y be two independent and positive random variables with respective density f_X and f_Y and let $g : (0, +\infty) \longrightarrow (0, +\infty)$, be a bounded Borel function. Find

$$\mathbb{E}\left(g\left(\frac{X}{Y}\right)|Y\right),$$

the conditional expectation of $g(X/Y)$ given Y and then infer that $V = X/Y$ has a density that you will identify.

9. Let X, Y, Z be random variables such that (X, Z) and (Y, Z) are identically distributed. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel function such that $f(X)$ is integrable.
(i) Show that $\mathbb{E}(f(X)|Z) = \mathbb{E}(f(Y)|Z), a.s.$
(ii) Let $T_1, T_2, \ldots T_n$ be iid random variables with finite first moment and let $T = T_1 + \cdots + T_n$. Using (i) show that

$$\mathbb{E}(T_1|T) = \frac{T}{n}.$$

which shows that $\mathbb{E}(T_1|T) = T/n$.